M321 1,2 and 3

THE OPEN UNIVERSITY

Mathematics: A Third Level Course

Partial Differential Equations of Applied Mathematics Units 1,2 and 3

The Wave Equation
Classification and Characteristics
Elliptic and Parabolic Equations

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THE WAVE EQUATION
CLASSIFICATION AND CHARACTERISTICS
ELLIPTIC AND PARABOLIC EQUATIONS

Prepared by the Course Team

The Open University Press

Cover Illustration
The cover of this volume is illustrated by photographs of a soap film on a metal frame. The configuration of the film is given by a solution to Laplace's equation with suitable boundary conditions. Laplace's equation is discussed in *Unit 3, Elliptic and Parabolic Equations*.

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1.2

Unit 1 The Wave Equation

Contents

Set Books

G. D. Smith, *Numerical Solution of Partial Differential Equations* (Oxford, 1971).

H. F. Weinberger, *A First Course in Partial Differential Equations* (Blaisdell, 1965).

It is essential to have these books; the course is based on them and will not make sense without them. They are referred to in the text as S and W respectively.

Unit 1 is based on W: Chapter I, Sections 1, 2 and 4.

Conventions

Before working through this text, make sure you have read *A Guide to the Course: Partial Differential Equations of Applied Mathematics*.

References to Open University courses in mathematics take the form:

Unit M100 13, Integration II for the Mathematics Foundation Course,

Unit M201 23, The Wave Equation for the Linear Mathematics Course.

1.0 INTRODUCTION

The idea of wave motion, a periodic phenomenon in time and space, must be familiar to everyone. In many instances it can be observed by the unaided eye as in the case of waves on the surface of water or the movement of a disturbance along a rope after the end has been moved suddenly. In other cases, the motion of individual waves is impossible to observe directly, as in the case of vibrations of a very taut violin string or sound waves in the air. Only after detailed experiment and analysis can it be shown that the phenomenon is a type of wave motion. One feature common to all these types of wave motion is that energy is transferred through the medium without permanently displacing it. Even if no medium is present, such as in the case of electromagnetic waves *in vacuo*, energy is still transferred.

In the physical world as we understand it there are three spatial dimensions and one time dimension. Variables describing physical quantities which vary continuously in space and time are, in general, functions of all four of the coordinates necessary to specify a point in the four-dimensional vector space making up the space–time continuum.

The use of all four coordinates to analyse the wave motion tends to obscure the basic relation between the time and space variations which is characteristic of the motion. We therefore adopt the usual approach and investigate firstly one-dimensional wave motion in which only one spatial coordinate is used.

You have met the one-dimensional homogeneous wave equation in *Unit M201 23, The Wave Equation*. To refresh your memory, it is the equation

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0,$$

where u is a function of two variables and c is a constant. Any function u satisfying this equation is a solution of the wave equation with parameter c.

The question of suitably defining a domain for u and the need for imposing further conditions on u before a unique solution can be obtained will be discussed in this unit.

In Section 1.1, we start by deriving the wave equation for three different physical situations in order to illustrate the importance of the equation in applied mathematics and to indicate some of the methods which could be used to derive the wave equation in other situations not discussed in this text. In Section 1.2 we discuss a method of solving the wave equation.

1.1 PHYSICAL EXAMPLES OF THE WAVE EQUATION

1.1.0 Introduction

You will recall from *Unit M201 23* that the wave equation arises when a mathematical modelling process is applied to:

(a) the longitudinal vibrations of a heavy spring;

(b) the current and voltage oscillations in an electrical transmission line.

There is no need for you to re-read the relevant sections of *Unit M201 23* in detail but a brief glance at them may refresh your memory and help with what follows in this unit.

In the next three sections we are going to apply the processes of mathematical modelling to a further three physical situations and demonstrate that the wave equation provides an approximate mathematical model of the situation in each case. These are:

(a) the transverse vibrations of a stretched string;

(b) the longitudinal vibrations of an elastic bar;

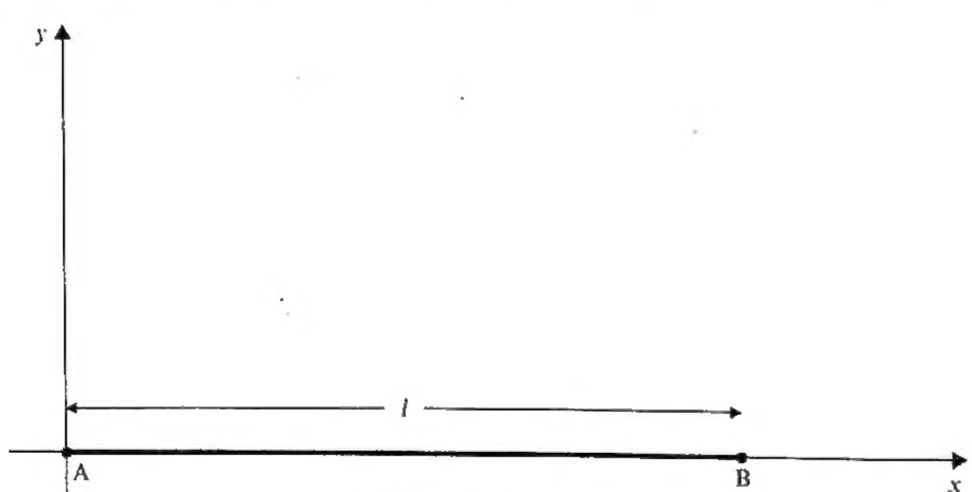
(c) one-dimensional sound waves in a gas.

The wave equation also arises from other physical situations, many of which require a wider and deeper understanding of physics than we assume for this course. We hope that the few examples we do provide are sufficient to highlight the importance of the wave equation in applied mathematics.

1.1.1 The Transverse Vibrations of a Stretched String

If a string is stretched and the ends attached to fixed supports A and B, it is intuitively obvious that a possible state of the string is that of equilibrium, i.e. each particle of the string lies on the straight line between the supports and is in the same position at all times.

To study the non-equilibrium positions, we use a rectangular coordinate system (x, y) for the plane so that support A lies at the origin and support B on the x-axis at $x = l$.



String equilibrium position

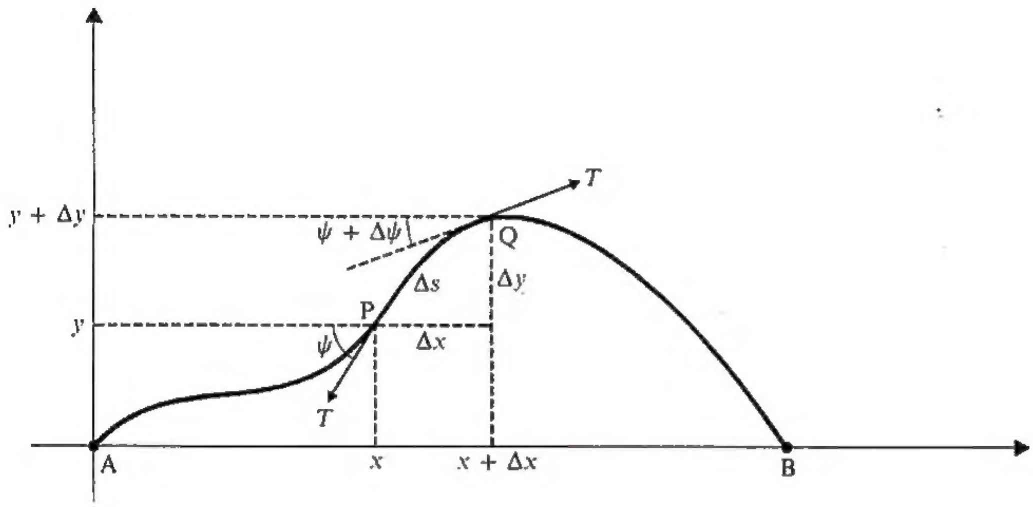
A particle of the string can be displaced from its equilibrium position longitudinally (parallel to the x-axis) and/or transversely (parallel to the y-axis). We consider only transverse displacements. If a segment of string is displaced transversely from its equilibrium position it exerts forces on the neighbouring segments of string on either side. As a result, these are no longer in equilibrium. They too will move transversely

and so the effect of a transverse displacement is propagated in both directions along the string.

To look at the situation in more detail we set up a mathematical model of the string.

Let the string have uniform line density ρ (i.e. mass ρ per unit length). Let the tension in the string be T and let the transverse displacement of the particle of the string with coordinate x be $y(x, t)$ at time t. For small displacements $y(x, t)$ we may assume that T is constant. We consider the small element of string between points P and Q with coordinates (x, y) and $(x + \Delta x, y + \Delta y)$ respectively* at time t. Let the length of this element be Δs and let the tangent to the string make an angle ψ with the x-axis at the point P and $\psi + \Delta\psi$ at the point Q. We see that

$$\psi(x + \Delta x, t) = \psi(x, t) + \Delta\psi(x, t).$$



Now we require Newton's Second Law of Motion which you met in *Unit M100 31, Differential Equations II*. Briefly the law states that

FORCE = MASS × ACCELERATION.

We shall apply Newton's Second Law of Motion to the *transverse* motion of the element of string PQ.

A FORCE acting in the plane may be represented by a geometric vector $\mathbf{F}$ (see *Unit M100 22, Linear Algebra I*). If $\mathbf{i}$ and $\mathbf{j}$ are unit vectors in the x- and y-directions respectively, then $\{\mathbf{i}, \mathbf{j}\}$ is a basis for our space of geometric vectors. We then have

$$\mathbf{F} = F_x\mathbf{i} + F_y\mathbf{j},$$

where F_x and F_y are uniquely determined real numbers which we call the **x-component** and the **y-component** of the force $\mathbf{F}$, respectively. It should be clear that if $\mathbf{F}$ represents a force whose magnitude is F and whose direction makes an angle θ with the positive direction of the x-axis when measured counterclockwise then

$$F_x = F\cos\theta$$

and

$$F_y = F\sin\theta.$$

So the y-component of the total force acting on the element of string PQ is

$$T\sin(\psi + \Delta\psi) - T\sin\psi.$$

The MASS of PQ is $\rho\Delta s$, and since

$$\Delta s \simeq \text{length of the } \textit{chord} \text{ PQ} = \Delta x \sec\psi$$

this mass approximately equals $\rho\Delta x \sec\psi$.

* In this course, as in the Linear Mathematics Course, we shall use the simplifying convention of denoting the image under a function by the symbol for the function in circumstances where there is no confusion.

Finally, the ACCELERATION of PQ in the y-direction at time t is given by the second derivative of the function

$$t \longmapsto y(x, t) \quad t \geqslant 0,$$

which is just

$$\frac{\partial^2 y}{\partial t^2}(x, t).$$

Collecting the pieces and applying Newton's Second Law of Motion, we obtain

$$T[\sin(\psi + \Delta\psi) - \sin\psi] = \rho\Delta x \sec\psi \times \frac{\partial^2 y}{\partial t^2}.$$

We shall now proceed to the limit as Δx and $\Delta\psi$ tend to zero. We have

$$\lim \frac{\sin(\psi + \Delta\psi) - \sin\psi}{\Delta\psi} = \frac{d}{d\psi}(\sin\psi)$$
$$= \cos\psi,$$

and

$$\lim \frac{\Delta\psi(x, t)}{\Delta x} = \lim \frac{\psi(x + \Delta x, t) - \psi(x, t)}{\Delta x}$$
$$= \frac{\partial\psi}{\partial x}(x, t).$$

This gives us

$$T\cos^2\psi \frac{\partial\psi}{\partial x} = \rho \frac{\partial^2 y}{\partial t^2}. \tag{1}$$

We know that at any instant of time

$$\tan\psi(x, t) = \frac{\partial y}{\partial x}(x, t)$$

by the definition of the tangent (*Unit M100 12, Differentiation I*). Therefore

$$\frac{\partial}{\partial x}(\tan\psi) = \frac{\partial^2 y}{\partial x^2},$$

that is,

$$\sec^2\psi \frac{\partial\psi}{\partial x} = \frac{\partial^2 y}{\partial x^2}.$$

Substituting the last result into Equation (1) leads to

$$T\cos^4\psi \frac{\partial^2 y}{\partial x^2} = \rho \frac{\partial^2 y}{\partial t^2}. \tag{2}$$

Finally,

$$\cos^4\psi = \frac{1}{\sec^4\psi} = \frac{1}{(1 + \tan^2\psi)^2} = \frac{1}{[1 + (\partial y/\partial x)^2]^2}.$$

If the displacements $y(x, t)$ are small compared with the length of the string and y is a fairly smooth function of x, then $(\partial y/\partial x)^2 \ll 1$ and $\cos^4\psi \simeq 1$. (The symbol $\ll$ means "very much smaller than".) This assumption is equivalent to the assumption that ψ remains small. Equation (2) becomes, approximately,

$$\frac{\partial^2 y}{\partial t^2} - \frac{T}{\rho}\frac{\partial^2 y}{\partial x^2} = 0,$$

or

$$\frac{\partial^2 y}{\partial t^2} - c^2 \frac{\partial^2 y}{\partial x^2} = 0, \tag{3}$$

where $c^2 = T/\rho$.

Equation (3) is the **wave equation in one dimension**. It describes the transverse motion of a stretched string quite well provided that the displacement $y(x, t)$ is not too large. Note that it is a *linear* differential equation. Thus, if y_1 and y_2 are any two solutions, then $A_1 y_1 + A_2 y_2$ is a solution for all choices of the constants A_1 and A_2.

In addition to satisfying the wave equation, the transverse displacement $y(x, t)$ must satisfy various *initial* and *boundary* conditions. At the fixed supports A and B on the x-axis, where $x = 0$ and $x = l$ respectively, we have $y(0, t) = y(l, t) = 0$ throughout the motion. These conditions are a particular form of **boundary conditions**, i.e. conditions at a spatial boundary for all times. In addition, at the time $t = 0$ we can start the motion of the string with a given shape

$$y(x, 0) = f(x)$$

and a given velocity distribution

$$\frac{\partial y}{\partial t}(x, 0) = g(x).$$

These conditions are a particular form of **initial conditions**.

We shall see in Section 1.2 that these initial conditions and boundary conditions constitute a set of conditions that determines uniquely the solution to the wave equation for the string for all future time along the whole length of string. This problem, formulated mathematically as follows, is the **vibrating string problem**

$$\frac{\partial^2 y}{\partial t^2} - c^2 \frac{\partial^2 y}{\partial x^2} = 0 \qquad (0 < x < l, t > 0)$$

$$y(x, 0) = f(x) \qquad (0 \leqslant x \leqslant l)$$

$$\frac{\partial y}{\partial t}(x, 0) = g(x) \qquad (0 \leqslant x \leqslant l)$$

$$y(0, t) = 0 \qquad (t \geqslant 0)$$

$$y(l, t) = 0 \qquad (t \geqslant 0)$$

Find a function y satisfying the equations above and continuous in the domain $[0, l] \times [0, \infty)$.

In *W*: *pages 1 to 5* the vibrating string problem is derived in a more rigorous way and in more general circumstances. We feel that Weinberger's treatment is too complex and long for inclusion in this course and even then it does not tell the full story.

SAQ 1

If $\psi(x)$ is the angle between the tangent to the curve $y(x, t_0)$ at some fixed t_0 and the x-axis, and $s(x)$ is the length of the curve from some fixed point to the point (x, y), we can define the *curvature* $K(x)$ of the curve at the point (x, y) by the equation

$$K(x) = \frac{\psi'(x)}{s'(x)}.$$

Find K in terms of y' and y'', where y' is the derived function of $x \longmapsto y(x, t_0)$.

(Solution on p. 32.)

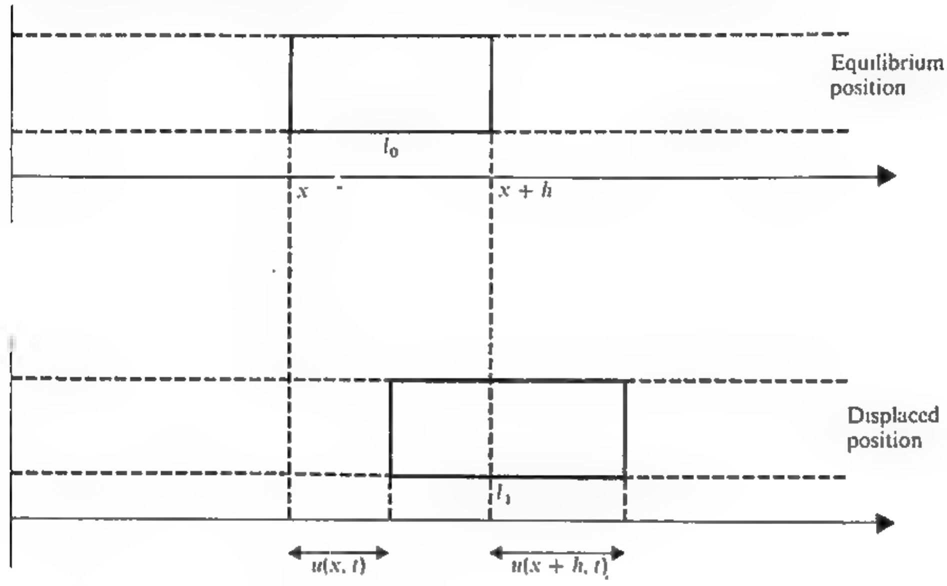
1.1.2 The Longitudinal Vibrations of an Elastic Bar

READ W: page 6, line 20 to page 7, line 10.

Notes

(i) *W: page 6, lines −13 to −7*
We show how $\partial u/\partial x(x, t)$ is to be interpreted as the elongation per unit length at the point x at time t.

Consider the element of the rod which lies between the planes x and $x + h$ in the equilibrium position, where h is small, as shown in the diagram.



The length of the element in the equilibrium position is

$$l_0 = (x + h) - x = h.$$

The length of the same element in the displaced position at time t is

$$\begin{aligned} l_1 &= [u(x + h, t) + (x + h)] - [u(x, t) + x] \\ &= u(x + h, t) - u(x, t) + h. \end{aligned}$$

Therefore the elongation of the element per unit length at time t is

$$\frac{l_1 - l_0}{l_0} = \frac{u(x + h, t) - u(x, t)}{h}.$$

We now take the limit as h becomes small to obtain the elongation per unit length at point x at time t:

$$\lim_{h \to 0} \frac{u(x + h, t) - u(x, t)}{h} = \frac{\partial u}{\partial x}(x, t).$$

We do not propose to justify the statement in W that the tension T depends linearly on $\partial u/\partial x$ (and not on higher derivatives of u). We shall adopt the view that

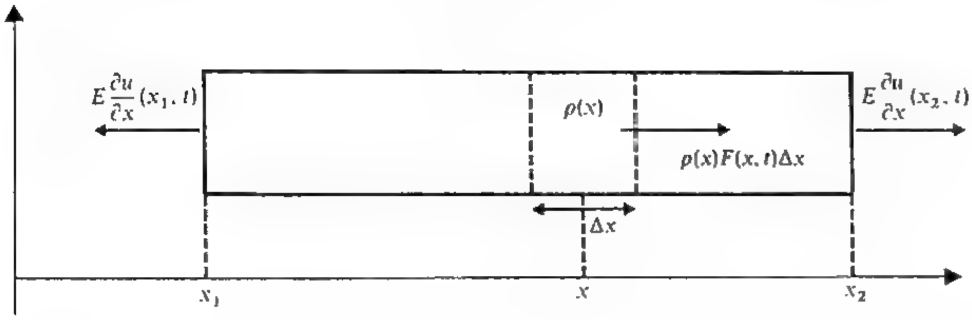
$$T(x, t) = E\frac{\partial u}{\partial x}(x, t)$$

is an empirical result (i.e. based on experimental evidence) which is valid provided $\partial u/\partial x$ remains small.

(ii) *W: page 6, line −6*
A body force is a force which acts on each element of a body from outside the body. The usual example is a gravitational force.

(iii) *W: page 6, line −4*
We can write this equation as

$$E\frac{\partial u}{\partial x}(x_2,t) - E\frac{\partial u}{\partial x}(x_1,t) + \int_{x_1}^{x_2} \rho(x)F(x,t)dx = \int_{x_1}^{x_2} \rho(x)\frac{\partial^2 u}{\partial t^2}(x,t)dx.$$



The left-hand side corresponds to the total force (per unit cross-sectional area) in the x-direction acting on the portion of the rod between the planes $x = x_1$ and $x = x_2$. The right-hand side is the product of mass (per unit cross-sectional area) and acceleration integrated over the same portion of the rod. It is the analogue, for a continuous medium, of

$$\sum_i m_i \frac{d^2 u_i}{dt^2}(t)$$

for a system of discrete masses m_i and corresponding displacements $u_i(t)$. The continuous variable x replaces the discrete variable i as a labelling parameter.

The equation represents the statement of Newton's Second Law of Motion for our particular situation.

(iv) *W: page 6, line −2*
This equation comes from the previous one by means of the Fundamental Theorem of Calculus (*Unit M100 13, Integration II*), applied to the function which maps

$$x_2 \longmapsto \int_{x_1}^{x_2} \rho\left[\frac{\partial^2 u}{\partial t^2} - F\right].$$

We can write the result as

$$\frac{\partial^2 u}{\partial t^2} - c^2\frac{\partial^2 u}{\partial x^2} = F, \qquad \text{with } c = \sqrt{\frac{E}{\rho}}.$$

Note that c is a function which is constant only if the mass density $\rho(x)$ does not vary with x. If F is not the zero function, the above equation is **nonhomogeneous**; if

$$F: x \longmapsto 0 \qquad 0 \leqslant x \leqslant l,$$

the equation is **homogeneous**.

(v) *W: page 7, line 1*
This sentence refers to boundary conditions, which were discussed briefly in Section 1.1.1. For our problem they are dealt with in SAQ 3 and SAQ 4.

(vi) *W: page 7, line 2*
The equilibrium solution referred to here has a constant tension T_0 associated with it. It is a different equilibrium solution from that used in the definition of the displacement $u(x, t)$. The latter is the equilibrium solution with zero tension.

SAQ 2

Suppose that, in the analysis of the motion of the elastic bar, we replace the relation

$$T = E\frac{\partial u}{\partial x}$$

by $$T = E\frac{\partial u}{\partial x} + K\left(\frac{\partial u}{\partial x}\right)^2.$$

where K is constant throughout the motion. Find the equation which replaces the nonhomogeneous wave equation

$$E\frac{\partial^2 u}{\partial x^2} - \rho\frac{\partial^2 u}{\partial t^2} = -\rho F.$$

Is it linear?

(Solution on p. 32.)

SAQ 3

An elastic bar, bounded by the planes $x = 0$ and $x = l$, has its ends fixed absolutely rigidly to a table top made of incompressible material. What are the boundary conditions appropriate to the longitudinal vibration problem of the bar?

(Solution on p. 32.)

SAQ 4

An elastic bar, bounded by the planes $x = 0$ and $x = l$ in the equilibrium position, has free ends. What are the boundary conditions appropriate to its longitudinal vibrations?

(Solution on p. 33.)

1.1.3 One-dimensional Sound Waves in a Gas

The final example of wave motion that we consider is sound waves. Sound waves differ from waves along a string in that they are longitudinal waves, as in the vibration of the bar studied in the previous section. The molecules of the air move in the direction of propagation of the wave, so that there are alternate compressions and rarefactions of the gas, and the restoring force responsible for propagating the wave is simply the opposition of the gas to compression.

In this example we shall consider only plane waves, or waves having the same direction of propagation everywhere in space; waves inside tubes of uniform cross section will usually be plane waves, also sound waves a long way from the source will approximate to plane waves. We let x be the coordinate parallel to the direction of propagation.

To obtain the relevant equations of motion of the gas we first find a more general equation of the motion of an incompressible fluid in one dimension and then make the approximation that the motion departs only slightly from the equilibrium motion.

In order to describe the one-dimensional motion of a small disturbance (sound wave) in a gas we shall require, in addition to Newton's Second Law of Motion, an important physical law which we have not yet encountered. This is the Law of Conservation of Mass, which states that

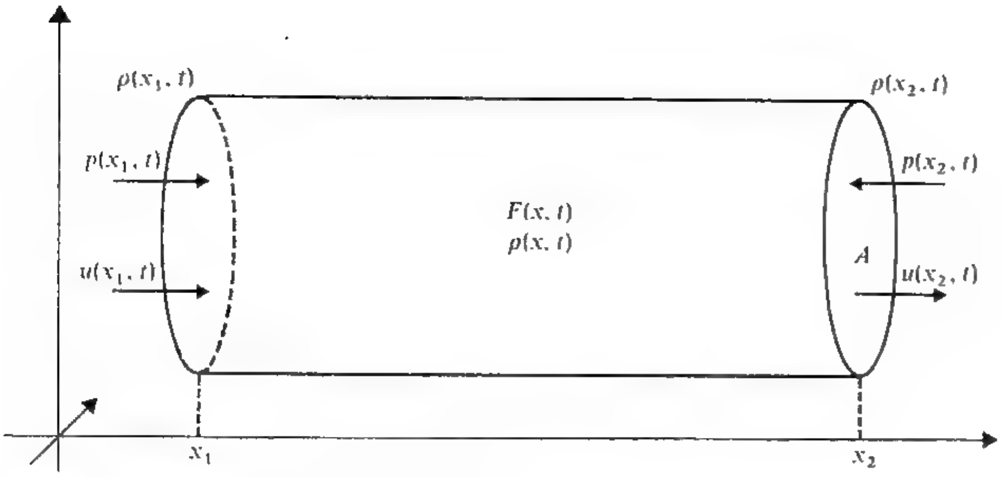
> the total mass of a quantity of material remains constant in time, however it becomes distributed in space.

Let $x = x_1$ be a fixed plane and (for $x_2 > x_1$) denote the total mass of gas between the planes $x = x_1$ and $x = x_2$ at time t by $M(x_1, x_2, t)$, as shown in the figure.

Let $u(x, t)$ be the velocity in the x-direction at time t of the gas passing through the plane $x =$ constant.

If $\rho(x, t)$ is the density (mass per unit volume), then gas enters the cylinder bounded by the planes $x = x_1$ and $x = x_2$ $(x_2 > x_1)$ at the rate $Au(x_1, t)\rho(x_1, t)$ and leaves it at the rate $Au(x_2, t)\rho(x_2, t)$, where A is the cross-sectional area of the cylinder. The net rate of increase of mass in the cylinder is $\partial M/\partial t(x_1, x_2, t)$. Therefore

$$\frac{\partial M}{\partial t}(x_1, x, t) = A[u(x_1, t)\rho(x_1, t) - u(x, t)\rho(x, t)] \qquad \forall x > x_1.$$



From the definitions of M and ρ we have

$$\frac{\partial M}{\partial x}(x_1, x, t) = A\rho(x, t).$$

Now we differentiate the expressions for $\partial M/\partial t$ and $\partial M/\partial x$ partially with respect to x and t respectively, thus obtaining

$$\frac{\partial^2 M}{\partial x \partial t} = -A\frac{\partial[u\rho]}{\partial x}$$

and

$$\frac{\partial^2 M}{\partial t \partial x} = A\frac{\partial \rho}{\partial t}.$$

Assuming that M is sufficiently well-behaved*, we obtain

$$\frac{\partial^2 M}{\partial x \partial t} = \frac{\partial^2 M}{\partial t \partial x}$$

which leads to

$$-\frac{\partial[u\rho]}{\partial x} = \frac{\partial \rho}{\partial t}$$

or

$$\rho\frac{\partial u}{\partial x} + u\frac{\partial \rho}{\partial x} + \frac{\partial \rho}{\partial t} = 0.$$

This is the *mass-conservation equation*, which represents mathematically the application to our problem of the Law of Conservation of Mass.

We now consider the application of Newton's Second Law of Motion to the gas in the cylinder $x_1 < x < x_2$.

To obtain the correct expression for the acceleration we introduce the concept of differentiation following the motion. Let $f(x, t)$ be some property of the gas, such as velocity or density. Then $\partial f/\partial t$ represents the rate of change at a *fixed* plane along the x-axis. The rate of change of f for a given element of the gas, whose position x varies with t, is given by the total derivative df/dt. We call this rate of change the **derivative following the motion.**

For a general variation of x and t we have the first-order Taylor approximation

$$\Delta f \simeq \frac{\partial f}{\partial x}\Delta x + \frac{\partial f}{\partial t}\Delta t$$

(*Unit M201 14, Bilinear and Quadratic Forms*). Thus

* See *Unit M201 14, Bilinear and Quadratic Forms*, Section 14.2.3.

$$\frac{\Delta f}{\Delta t} \simeq \frac{\partial f}{\partial x}\frac{\Delta x}{\Delta t} + \frac{\partial f}{\partial t},$$

and in the limit as Δf, Δx and Δt approach zero

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial t}$$

$$= u(x, t)\frac{\partial f}{\partial x} + \frac{\partial f}{\partial t}$$

where u gives the velocity.

This result is a particular example of the **chain rule for composite functions of several variables***, which is a generalization of the chain rule which you met in *Unit M100 12, Differentiation I.*

Suppose

$$w:(x, y, z) \longmapsto u,$$

and

$$\xi:(p, q) \longmapsto x,$$

$$\eta:(p, q) \longmapsto y,$$

$$\zeta:(p, q) \longmapsto z,$$

where (p, q) belongs to the domains of ξ, η, ζ. Then the chain rule tells us that

$$\frac{\partial u}{\partial p} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial p} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial p} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial p}$$

and

$$\frac{\partial u}{\partial q} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial q} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial q} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial q}.$$

Here $\partial u/\partial p$ means the derivative at $\bar{p} = p$ of the function

$$\bar{p} \longmapsto w(\xi(\bar{p}, q), \eta(\bar{p}, q), \zeta(\bar{p}, q)) \qquad (\bar{p} \in P);$$

$\partial u/\partial x$ means the derivative at $\bar{x} = x$ of the function

$$\bar{x} \longmapsto w(\bar{x}, y, z) \qquad (\bar{x} \in X);$$

and so on. (P and X are the relevant domains.)

The rule can be generalized to any number of domains and any number of compositions of functions. So, if

$$\alpha: t \longmapsto x,$$

$$\beta: t \longmapsto y,$$

$$\gamma: t \longmapsto z,$$

and

$$f:(x, y, z) \longmapsto v,$$

then we have

$$\frac{dv}{dt} = \frac{\partial v}{\partial x}\frac{dx}{dt} + \frac{\partial v}{\partial y}\frac{dy}{dt} + \frac{\partial v}{\partial z}\frac{dz}{dt},$$

judiciously choosing d or ∂ according to whether the relevant function has a single or multi-variable domain.

The acceleration of the element of gas which, at time t, occupies the disc (of the cylinder) at x is given by differentiating the velocity following the motion. Thus we obtain

* For a derivation of the chain rule for a function of several variables see a text on analysis, such as W. Kaplan, *Advanced Calculus* (Addison-Wesley, 1962).

$$a(x,t) = u(x,t)\frac{\partial u}{\partial x}(x,t) + \frac{\partial u}{\partial t}(x,t).$$

To apply Newton's Second Law of Motion we must specify the forces on the region of gas. The total force in the x-direction at time t on the cylinder of gas bounded by the planes $x = x_1$ and $x = x_2$ ($x_1 < x_2$) is composed of two different types of force. The first is due to the pressures (i.e. forces per unit area) across the ends of the cylinder, both of which are directed inwards: the second is the body force represented by the integral of the product of the density $\rho(x,t)$ and the body force per unit mass $F(x,t)$ over the whole of the cylinder. So the total force is given by

$$Ap(x_1,t) - Ap(x_2,t) + A\int_{x_1}^{x_2} \rho(x,t)F(x,t)dx,$$

where $p(x,t)$ is the pressure across the plane x = constant at time t. Newton's Second Law of Motion, applied to the total system, gives

$$\begin{aligned} p(x_1,t) - p(x_2,t) &+ \int_{x_1}^{x_2} \rho(x,t)F(x,t)dx \\ &= \int_{x_1}^{x_2} \rho(x,t)\left[u(x,t)\frac{\partial u}{\partial x}(x,t) + \frac{\partial u}{\partial t}(x,t)\right]dx \end{aligned}$$

after substituting the expression for the acceleration obtained previously, and dividing throughout by A. Partial differentiation of this equation with respect to x_2 gives

$$-\frac{\partial p}{\partial x}(x_2,t) + \rho(x_2,t)F(x_2,t) = \rho(x_2,t)\left[u(x_2,t)\frac{\partial u}{\partial x}(x_2,t) + \frac{\partial u}{\partial t}(x_2,t)\right],$$

or, in terms of functions,

$$\rho\frac{\partial u}{\partial t} + \rho u\frac{\partial u}{\partial x} + \frac{\partial p}{\partial x} = \rho F.$$

The two equations which we have derived here will be used in the next reading passage to obtain the equation for sound waves in a gas.

READ W: page 7, line 11 to page 8, line 7.

Notes

(i) *W: page 7, lines −17 and −16*

The pressure at a point in a gas depends on the density and the temperature. In general the temperature may vary independently of the density. If the gas is vibrating so rapidly that there is little time for much heat to be transmitted through the medium in one direction before the flow of heat is reversed, then the system is *adiabatic*. In these circumstances and with an initially uniform temperature distribution it can be shown that the pressure is completely determined by the density. (Note how Weinberger expresses this relationship by the applied mathematician's traditional abuse of notation: $p = p(\rho)$.)

For an ideal gas the relationship between p and ρ in the adiabatic situation is given by

$$p(x,t) = \alpha[\rho(x,t)]^{\gamma},$$

where α is a positive constant, which may be determined from a knowledge of the pressure and density at some (x,t), and γ is a constant, greater than 1, which depends on the molecular structure of the gas and certain other factors.

(ii) *W: page 7, line −12*

The term ρ/ρ_0 is the ratio of two quantities of the same physical type: both are densities. It is therefore independent of the size of the unit we choose for the measurement of density. Such quantities are *dimensionless*—they are just numbers. Clearly $p'(\rho)/p'(\rho_0)$ is also dimensionless. It is not, however, quite so obvious that $u/[p'(\rho_0)]^{\frac{1}{2}}$ is dimensionless; we shall show that it is.

Let M, L and T denote the fundamental physical quantities *mass*, *length* and *time* respectively. *Velocity* has the dimensions LT^{-1}, and *force* has the dimensions MLT^{-2}, as we saw in *Unit M100 3, Operations and Morphisms*. **Pressure** is a force per unit area and so

$$\dim(\textit{pressure}) = \frac{\dim(\textit{force})}{\dim(\textit{area})} = \frac{MLT^{-2}}{L^2} = ML^{-1}T^{-2},$$

where dim is the function which maps a given physical quantity to its dimensions. Similarly,

$$\dim(\textit{density}) = \frac{\dim(\textit{mass})}{\dim(\textit{volume})} = ML^{-3}.$$

Thus dim maps

$$u \longmapsto LT^{-1}$$

and

$$\frac{dp}{d\rho} \longmapsto ML^{-1}T^{-2}/ML^{-3} = L^2T^{-2};$$

so that

$$\left(\frac{dp}{d\rho}\right)^{\frac{1}{2}} \longmapsto LT^{-1}.$$

So we see that u and $[p'(\rho_0)]^{\frac{1}{2}}$ have the same dimensions and therefore $u/[p'(\rho_0)]^{\frac{1}{2}}$ is dimensionless.

It is meaningful in physics to say only that *dimensionless* quantities are small (or large). The sizes of other quantities can be changed arbitrarily by changing the units in terms of which we measure them.

(iii) *W: page 7, lines −8 and −7*

Equations (1.14) are derived from Equations (1.13) as follows.

Let

$$\bar{\rho}(x,t) = \frac{\rho(x,t)}{\rho_0} - 1,$$

$$\bar{u}(x,t) = \frac{u(x,t)}{\sqrt{p'(\rho_0)}},$$

and assume that $\bar{\rho}(x,t)$ and $\bar{u}(x,t)$ (which are dimensionless) are small compared with 1. Substituting for ρ and u in the first of Equations (1.13), we obtain

$$\rho_0\sqrt{p'(\rho_0)}\left[\frac{\partial \bar{u}}{\partial x} + \bar{\rho}\frac{\partial \bar{u}}{\partial x} + \bar{u}\frac{\partial \bar{\rho}}{\partial x}\right] + \rho_0\frac{\partial \bar{\rho}}{\partial t} = 0.$$

The second and third terms are small and may be neglected; this leads to

$$\rho_0\sqrt{p'(\rho_0)}\frac{\partial \bar{u}}{\partial x} + \rho_0\frac{\partial \bar{\rho}}{\partial t} = 0.$$

Expressed in terms of ρ and u, this equation becomes

$$\rho_0\frac{\partial u}{\partial x} + \frac{\partial \rho}{\partial t} = 0,$$

which is the first of Equations (1.14).

The second of Equations (1.13) may be approximated similarly by

$$\rho_0\frac{\partial u}{\partial t} + \frac{\partial p}{\partial x} = \rho_0 F.$$

By the chain rule for differentiation, we have

$$\frac{\partial p}{\partial x} = p'(\rho)\frac{\partial \rho}{\partial x}.$$

Put

$$p'(\rho) = p'(\rho_0)(1 + \alpha),$$

where α is assumed small compared with 1. Neglecting α, this leads to

$$\rho_0\frac{\partial u}{\partial t} + p'(\rho_0)\frac{\partial \rho}{\partial x} = \rho_0 F.$$

This is the second of Equations (1.14).

(iv) *W: page 7, line −4*
We need to assume that

$$\frac{\partial^2 \rho}{\partial t \partial x} = \frac{\partial^2 \rho}{\partial x \partial t}.$$

(v) *W: page 8, line 1*
We need to assume that

$$\frac{\partial^2 u}{\partial x \partial t} = \frac{\partial^2 u}{\partial t \partial x}.$$

SAQ 5

In the adiabatic sound wave problem just discussed, find the forms of the wave equation obeyed by u and ρ if the gas is an ideal gas (i.e. $p = \alpha\rho^\gamma$) and the body force is given by

$$F(x, t) = F_0 \sin ax \cos bt.$$

(Solution on p. 33.)

SAQ 6

In the one-dimensional gas flow problem, if the velocity $u(x, t)$ is independent of the time coordinate t, is the acceleration necessarily zero?

(Solution on p. 34.)

SAQ 7

For a steady one-dimensional gas flow (i.e. $u(x, t)$, $\rho(x, t)$ and $p(x, t)$ are independent of time) in which the body force is zero, prove that

$$\rho u = K$$

and

$$p = -Ku + L,$$

where K and L are constant functions.

(Solution on p. 34.)

1.2 SOLVING THE HOMOGENEOUS WAVE EQUATION

1.2.0 Introduction

We now show how to solve the wave equation in terms of the initial conditions over the whole future domain for two different types of boundary conditions.

If

$$u|_{x=a} = 0 \qquad (t \geqslant 0)$$

for $a = 0$ or $a = l$, then we have a **fixed boundary condition** at $x = a$. If

$$\left.\frac{\partial u}{\partial x}\right|_{x=a} = 0 \qquad (t \geqslant 0)$$

for $a = 0$ or $a = l$, then we have a **free boundary condition** at $x = a$. We have already met physical situations from which these terms arise, in Section 1.1.

We shall first solve the wave equation

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0 \qquad (0 < x < l, t > 0)$$

subject to the initial conditions

$$u|_{t=0} = f(x) \qquad (0 \leqslant x \leqslant l)$$

and

$$\left.\frac{\partial u}{\partial t}\right|_{t=0} = g(x) \qquad (0 \leqslant x \leqslant l)$$

using d'Alembert's method of solution which was treated briefly in *Unit M201 23*.

Note that, to avoid confusion, we are restricting u to be a variable; we shall use v for the function

$$(x, t) \longmapsto u$$

to preserve a distinction between a function and its value. This will be useful when we wish to use another set of coordinates to define a function

$$w : (\xi, \eta) \longmapsto u.$$

1.2.1 The Fixed Boundary Problem

READ W: Section 2, page 8 to page 15, line 6.

The first part of this reading passage (as far as *W: page 10, line −5*) is d'Alembert's solution of the wave equation with given initial conditions. The remainder of the passage determines the solution of the problem posed by including fixed boundary conditions.

Notes

(i) *W: page 9, line 8 to Equation (2.3)*
The change of variables in this passage is most easily followed if we carefully distinguish the functions involved. Let

$$\alpha(x, t) = \xi$$

and

$$\beta(x, t) = \eta;$$

we have already, in the Introduction, defined the functions

$$v:(x,t) \longmapsto u$$

and

$$w:(\xi,\eta) \longmapsto u.$$

Thus

$$w:(\alpha(x,t),\beta(x,t)) \longmapsto u.$$

Applying the chain rule for a function of several variables to the composite function $w \circ (\alpha, \beta)$, we obtain

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial x}.$$

Suppose now that the functions α and β are given by

$$\alpha:(x,t) \longmapsto x + ct,$$

$$\beta:(x,t) \longmapsto x - ct;$$

then

$$\frac{\partial \xi}{\partial x} = 1, \quad \frac{\partial \eta}{\partial x} = 1.$$

So

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial \xi} + \frac{\partial u}{\partial \eta}.$$

A further application of the chain rule yields

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial(\partial u/\partial x)}{\partial x} = \frac{\partial^2 u}{\partial \xi^2} + \frac{\partial^2 u}{\partial \xi \partial \eta} + \frac{\partial^2 u}{\partial \eta \partial \xi} + \frac{\partial^2 u}{\partial \eta^2}.$$

Similarly, we may obtain

$$\frac{\partial^2 u}{\partial t^2} = c^2\left[\frac{\partial^2 u}{\partial \xi^2} - \frac{\partial^2 u}{\partial \xi \partial \eta} - \frac{\partial^2 u}{\partial \eta \partial \xi} + \frac{\partial^2 u}{\partial \eta^2}\right].$$

If we assume that

$$\frac{\partial^2 u}{\partial \xi \partial \eta} = \frac{\partial^2 u}{\partial \eta \partial \xi},$$

the equation

$$\frac{\partial^2 u}{\partial t^2} - c^2\frac{\partial^2 u}{\partial x^2} = 0$$

becomes

$$-4c^2\frac{\partial^2 u}{\partial \xi \partial \eta} = 0.$$

Since $c \neq 0$ (by assumption) we obtain, expressing u in terms of the function w,

$$\frac{\partial^2 w}{\partial \xi \partial \eta}(\xi,\eta) = 0.$$

Integration with respect to ξ yields

$$\frac{\partial w}{\partial \eta}(\xi,\eta) = r(\eta),$$

where r is an arbitrary function. Further integration, this time with respect to η, yields

$$w(\xi,\eta) = \int_a^\eta r(\bar{\eta})d\bar{\eta} + p(\xi),$$

where a is a constant and p is another arbitrary function. If q denotes an integral of r, we obtain

$$\begin{aligned} u &= w(\xi, \eta) \\ &= p(\xi) + q(\eta) \end{aligned}$$

where p and q are arbitrary functions. In terms of x and t we have, therefore,

$$u = p(x + ct) + q(x - ct).$$

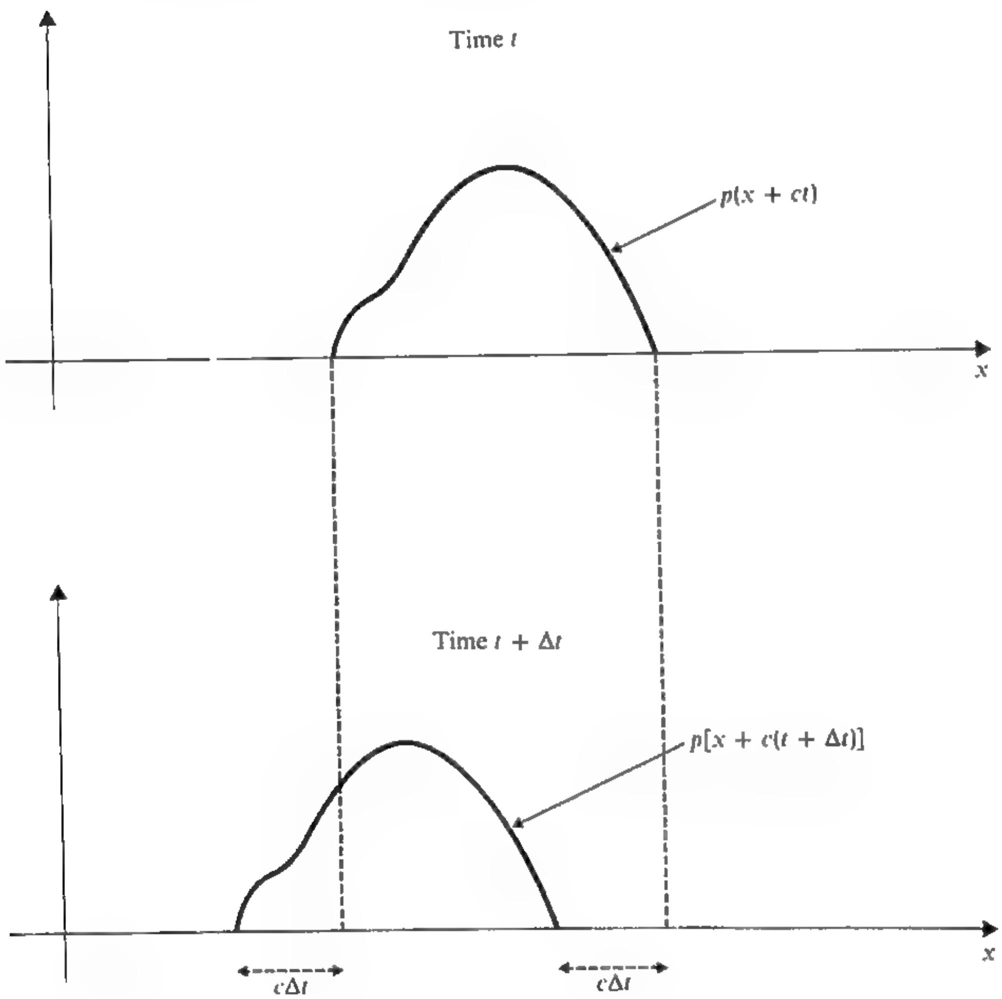
(ii) *W: page 9, line −4 to page 10, line 2*
The travelling wave nature of the functions

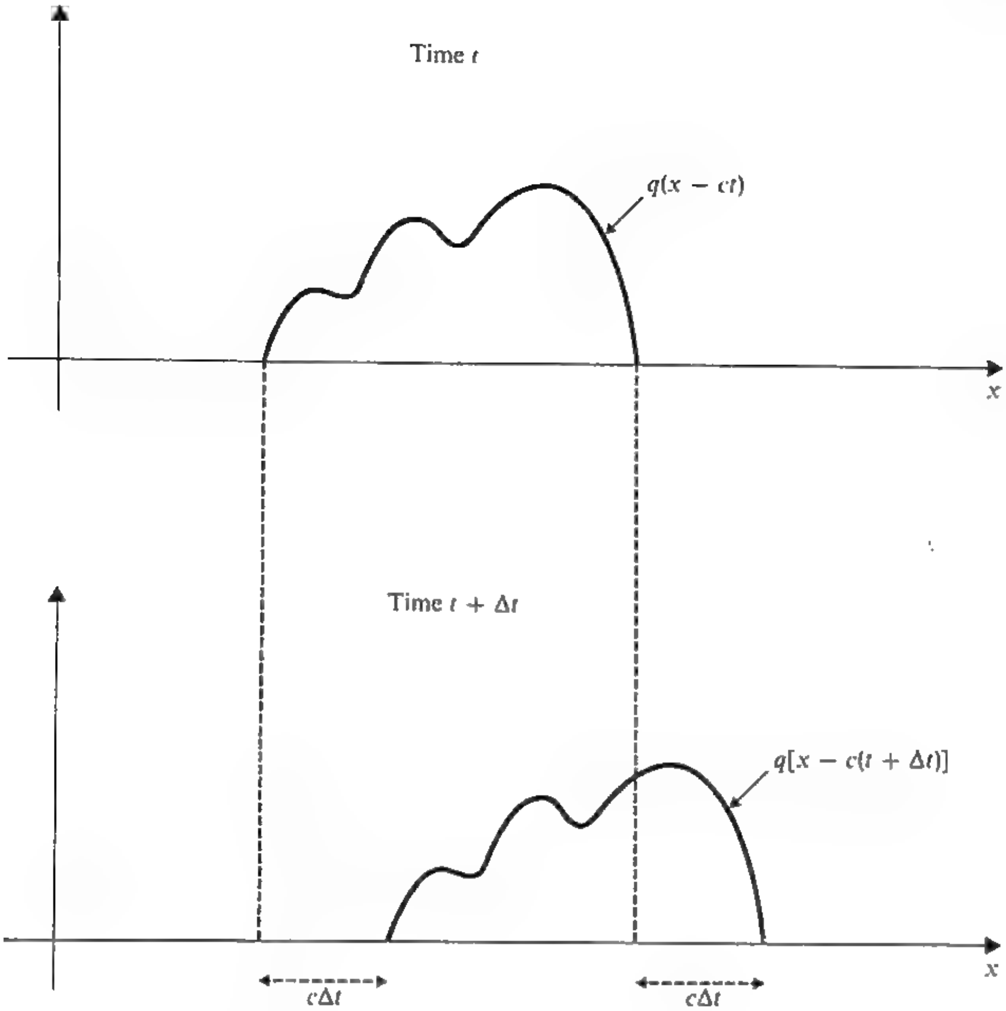
$$(x, t) \longmapsto p(x + ct)$$

and

$$(x, t) \longmapsto q(x - ct)$$

is illustrated by the following diagrams.





(iii) *W: page 10, lines −10 to −8*
The inequalities

$$0 \leqslant x + ct \leqslant l$$

$$0 \leqslant x - ct \leqslant l$$

are, when expressed singly,

$$x + ct \geqslant 0,$$

$$x + ct \leqslant l,$$

$$x - ct \geqslant 0,$$

$$x - ct \leqslant l.$$

The third of these is equivalent to

$$t \leqslant \frac{x}{c}.$$

The second is equivalent to

$$t \leqslant \frac{l - x}{c}.$$

We have the basic restriction

$$t \geqslant 0$$

as the wave equation is only being considered in the domain $t \geqslant 0$. Since c is defined to be positive and $t \geqslant 0$, it follows that

$$ct \geqslant 0.$$

Thus

$$x + ct \geqslant 0 \text{ is a consequence of } x - ct \geqslant 0,$$

and

$$x - ct \leqslant l \text{ is a consequence of } x + ct \leqslant l.$$

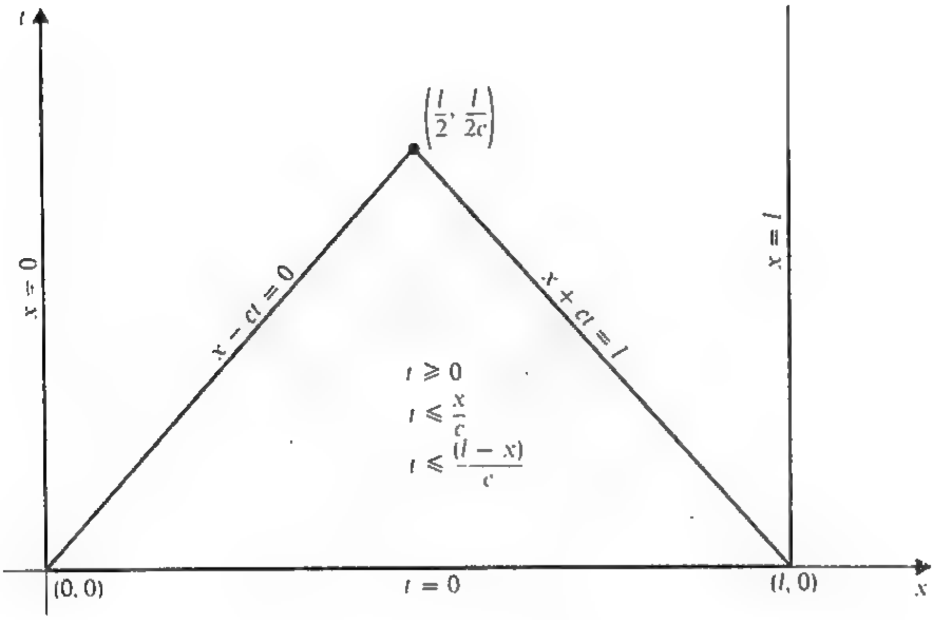
Therefore all four inequalities $0 \leqslant x + ct \leqslant l$ and $0 \leqslant x - ct \leqslant l$ are satisfied if

$$t \geqslant 0, t \leqslant \frac{x}{c} \quad \text{and} \quad t \leqslant \frac{l - x}{c}.$$

The triangular region

$$t \geqslant 0, t \leqslant \frac{x}{c}, t \leqslant \frac{l - x}{c}$$

is indicated by dark shading in the following diagram.



In this region the solution of the wave equation is given by

$$u = \tfrac{1}{2}[f(x + ct) + f(x - ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(\bar{x})d\bar{x}$$

and is thus dependent ONLY on the initial conditions (the conditions at $t = 0$ for u and $\partial u/\partial t$ for $0 \leqslant x \leqslant l$) and NOT on the boundary conditions (the conditions on, for instance, u at the ends of the string for all $t \geqslant 0$). Indeed, as far as the solution in the triangular region is concerned, the values of $v(0, t)$ and $v(l, t)$ could be given by any functions consistent with the initial conditions, and the solution in this region would be unchanged.

We are interested in the solution over the whole domain $t \geqslant 0$, $0 \leqslant x \leqslant l$ and not just in the triangular region of the diagram. In determining the solution over this larger domain we shall find that the boundary conditions are needed.

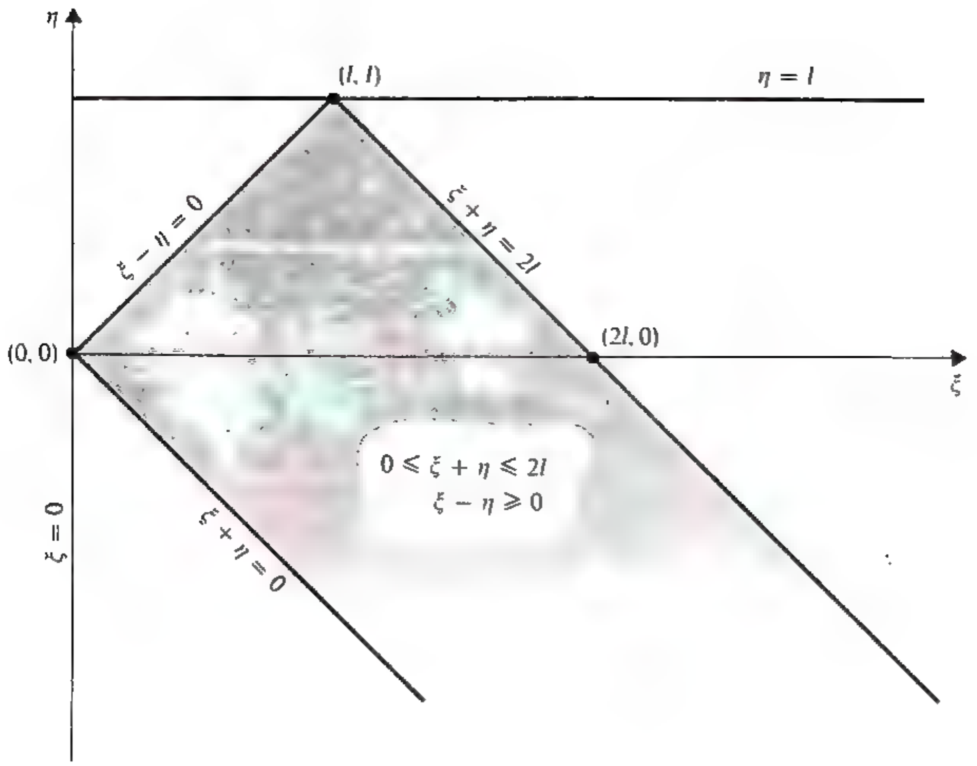
(iv) *W: page 11, lines 12 to 16*

We wish to obtain u over the domain defined by the inequalities

$$0 \leqslant x \leqslant l, \quad t \geqslant 0.$$

These inequalities are equivalent to

$$0 \leqslant \xi + \eta \leqslant 2l, \quad \xi - \eta \geqslant 0.$$



It is clear from the diagram that this domain is contained in the domain defined by the inequalities

$$\eta \leqslant l, \quad \xi \geqslant 0.$$

Thus, a knowledge of $p(\xi)$ for $\xi \geqslant 0$ and $q(\eta)$ for $\eta \leqslant l$ is more than sufficient for a complete solution to our problem.

The general results obtainable from Equations (2.9) and (2.10) (*W: page 11*) are:

$$\begin{aligned}
p(\xi) &= -q(2nl - \xi) && (2n-1)l \leqslant \xi \leqslant 2nl,\\
p(\xi) &= p(\xi - 2nl) && 2nl \leqslant \xi \leqslant (2n+1)l,\\
q(\eta) &= q(\eta + 2nl) && -2nl \leqslant \eta \leqslant -(2n-1)l,\\
q(\eta) &= -p(-2nl - \eta) && -(2n+1)l \leqslant \eta \leqslant -2nl,
\end{aligned}$$

for $n = 1, 2, 3, \ldots$.

These equations enable us to find p and q throughout their respective domains $\xi \geqslant 0$ and $\eta \leqslant l$, given $p(\xi)$ and $q(\eta)$ defined only over $0 \leqslant \xi \leqslant l$, $0 \leqslant \eta \leqslant l$ by Equations (2.5) and (2.6) (*W: page 10*).

It should be emphasized that the fixed boundary conditions

$$u|_{x=0} = u|_{x=l} = 0, \quad t \geqslant 0$$

are responsible for giving us these particular results for extending the solution into the whole of the required domain.

SAQ 8

W: page 17, Exercise 3
Assume that the ends remain fixed.

(Solution on p. 34.)

SAQ 9

For the initial-boundary value problem

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0 \qquad (0 < x < l, t > 0),$$

$$u|_{t=0} = a \sin \frac{\pi x}{l} \qquad (0 \leqslant x \leqslant l),$$

$$\left.\frac{\partial u}{\partial t}\right|_{t=0} = -b \sin \frac{\pi x}{l} \qquad (0 \leqslant x \leqslant l),$$

$$u|_{x=0} = u|_{x=l} = 0 \qquad (t \geqslant 0),$$

where a and b are constants, show that

$$u|_{t=t_n} = 0 \qquad n = 0, 1, 2, \ldots,$$

where

$$t_n = \frac{l}{\pi c}\left[\tan^{-1} \frac{\pi a c}{l b}\right] + \frac{nl}{c}.$$

Interpret this result physically.

(Solution on p. 35.)

So far we have considered examples of the wave equation problem in which *initial* conditions $u|_{t=0}$ and $\left.\frac{\partial u}{\partial t}\right|_{t=0}$ are given. It is possible to have other conditions: for illustrative purposes we choose as given $u|_{t=x}$ and $\left.\frac{\partial u}{\partial t}\right|_{t=x}$, i.e. the variables are specified on the line $t = x$ instead of the line $t = 0$. Such conditions could be imposed, in the vibrating string problem, by a "position- and velocity-setting device" which travels along the x-axis at unit velocity. In the following example the wave velocity c is assumed to be smaller than that of the position- and velocity-setting device, i.e. $c < 1$.

Example

W: page 17, Exercise 10

Solution

The general solution to the wave equation is:

$$u = p(x + ct) + q(x - ct) \qquad (0 \leqslant x \leqslant 1, \quad t \geqslant x).$$

Since

$$u|_{t=x} = f(x) \qquad (0 \leqslant x \leqslant 1),$$

we have

$$p(x(1 + c)) + q(x(1 - c)) = f(x) \qquad (0 \leqslant x \leqslant 1).$$

Since

$$\left.\frac{\partial u}{\partial t}\right|_{t=x} = 0 \qquad (0 \leqslant x \leqslant 1),$$

we have

$$cp'(x(1 + c)) - cq'(x(1 - c)) = 0 \qquad (0 \leqslant x \leqslant 1),$$

using the chain rule.

Integration of the last equation with respect to x yields

$$c\frac{p(x(1 + c))}{1 + c} - c\frac{q(x(1 - c))}{1 - c} = K \qquad (0 \leqslant x \leqslant 1),$$

where K is a constant.

Solving the two linear equations in $p(x(1+c))$ and $q(x(1-c))$, we obtain

$$p(x(1+c)) = \tfrac{1}{2}(1+c)f(x) + \frac{K(1-c^2)}{2c} \qquad (0 \leqslant x \leqslant 1),$$

and

$$q(x(1-c)) = \tfrac{1}{2}(1-c)f(x) - \frac{K(1-c^2)}{2c} \qquad (0 \leqslant x \leqslant 1).$$

We may choose $K = 0$, since p and q will be added to obtain u and the constant terms will cancel. Therefore

$$p(\xi) = \tfrac{1}{2}(1+c)f\left(\frac{\xi}{1+c}\right) \qquad (0 \leqslant \xi \leqslant 1+c)$$

and

$$q(\eta) = \tfrac{1}{2}(1-c)f\left(\frac{\eta}{1-c}\right) \qquad (0 \leqslant \eta \leqslant 1-c).$$

The boundary condition

$$u|_{x=0} = 0 \qquad (t \geqslant 0)$$

yields

$$p(ct) + q(-ct) = 0 \qquad (t \geqslant 0);$$

that is,

$$q(\eta) = -p(-\eta) \qquad (\eta \leqslant 0).$$

The boundary condition

$$u|_{x=1} = 0 \qquad (t \geqslant 1)$$

yields

$$p(1+ct) + q(1-ct) = 0 \qquad (t \geqslant 1);$$

that is,

$$p(\xi) = -q(2-\xi) \qquad (\xi \geqslant 1+c).$$

The two equations

$$q(\eta) = -p(-\eta) \qquad (\eta \leqslant 0)$$

$$p(\xi) = -q(2-\xi) \qquad (\xi \geqslant 1+c)$$

are sufficient to extend $p(\xi)$ and $q(\eta)$ as far as necessary, because the inequalities

$$t \geqslant x, \quad 0 \leqslant x \leqslant 1,$$

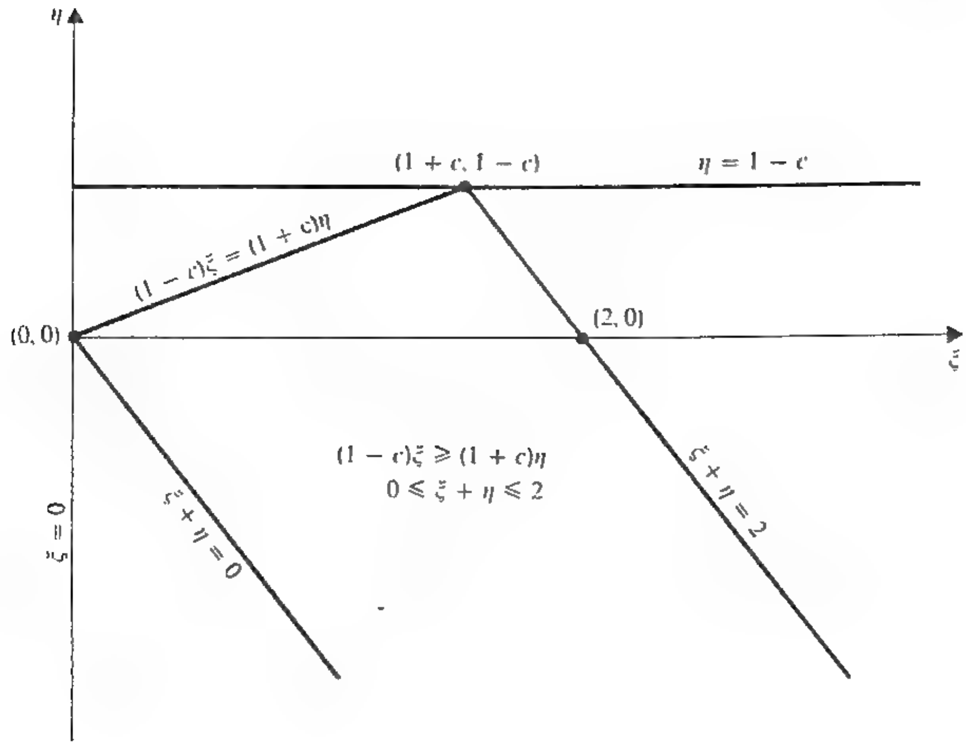
which define the domain over which the solution is required, are equivalent to the following inequalities:

$$(1-c)\xi \geqslant (1+c)\eta, \quad 0 \leqslant \xi + \eta \leqslant 2.$$

The region

$$\eta \leqslant 1-c, \quad \xi \geqslant 0,$$

over which we can determine $p(\xi)$ and $q(\eta)$, encloses the former domain and so the solution can be found. This is illustrated in the following diagram.



1.2.2 The Free Boundary Problem

So far we have concentrated our attention on fixed boundary conditions for the wave equation. In the next reading passage you will see how the combination of initial conditions and fixed boundary conditions can be regarded as a wholly initial value problem.

READ W: Section 4, page 21 to page 22, line −2.

In our derivation of the vibrating string problem (Section 1.1.1), if the end $x = 0$ of the string is constrained only in the x-direction but is unconstrained in the y-direction (for example, if the end of the string is attached to a massless ring which slides on a rod in the y-direction), then there is a *free boundary condition* at that end. Since there is no force on the string in the y-direction, the end of the string must remain perpendicular to the rod:

$$\frac{\partial y}{\partial x}(0, t) = 0.$$

In general a free boundary condition at $x = a$ is defined by

$$\left.\frac{\partial u}{\partial x}\right|_{x=a} = 0.$$

SAQ 10

A string of line density ρ and tension T, which in equilibrium lies along the x-axis with end-points $x = 0$ and $x = l$, has a small bead of mass m attached to the end of the string at $x = 0$. The bead is constrained to move along the line $x = 0$ but its y-coordinate is not constrained. What is the appropriate boundary condition at $x = 0$ for small transverse vibrations of the string?

(HINT: Use Newton's Second Law of Motion for the bead.)

(Solution on p. 36.)

SAQ 11

A string of line density ρ^- lies along the negative x-axis and another string of line density ρ^+ lies along the positive x-axis. They are joined at the point $x = 0$. The tension in the strings is T. If y^- and y^+ are the functions representing the transverse displacements on the negative and positive x-axes respectively, what are the boundary conditions applying at the point $x = 0$ when the junction is unconstrained in the y-direction?

(Solution on p. 37.)

In the next reading passage, we shall see how free boundary conditions can also be incorporated in the initial conditions in an appropriate way, enabling us to find solutions.

READ W: page 23, line 13 to page 24, line 12 (end of the section), omitting lines −6 to −3 on page 23.

The omitted paragraph relates to *characteristics* which we shall discuss in *Unit 2, Classification and Characteristics.*

General Comment

The type of boundary condition determines the choice of an odd or even extension of the functions f and g (the initial conditions). We use an odd extension for a fixed boundary condition and an even extension for a free boundary condition.

Example

W: page 24, Exercise 8

Solution

The function

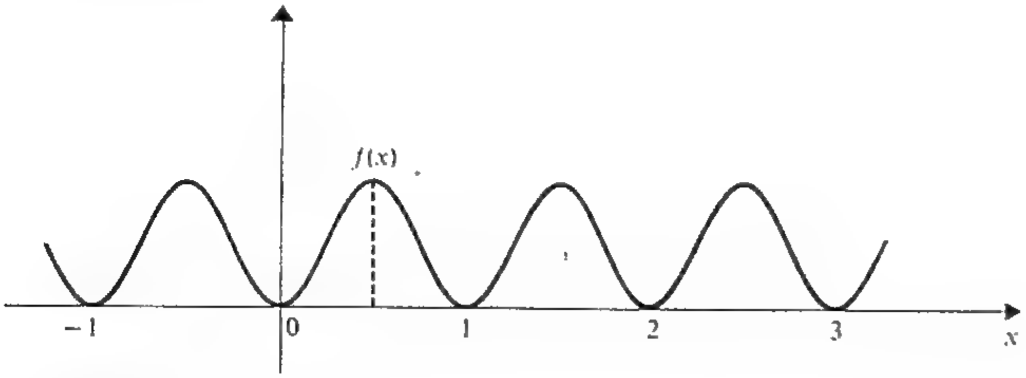
$$x \longmapsto x^2(1-x)^2 \qquad (x \in R)$$

is not even about either of the points $x = 0$ or $x = 1$, which is what we require for the extensions of f and g, since we have free boundary conditions at both $x = 0$ and $x = 1$.

The extension for f given by

$$f(x) = (x-n)^2[1-(x-n)]^2 \qquad (n \leqslant x \leqslant n+1, n = 0, \pm 1, \pm 2, \ldots)$$

is even about both $x = 0$ and $x = 1$. This enables us to extend the domain of f to the whole real line.



The function

$$x \longmapsto 1 \qquad (x \in R)$$

is even about both $x = 0$ and $x = 1$ and we shall therefore define g in that way for all $x \in R$. We have the general solution (Equation (2.16) in *W: page 13*)

$$u(x,t) = \tfrac{1}{2}[f(x+t) + f(x-t)] + \tfrac{1}{2}\int_{x-t}^{x+t} g(\bar{x})d\bar{x},$$

since $c = 1$. Substituting $g(\bar{x}) = 1$ and performing the integration, we obtain

$$u(x, t) = \tfrac{1}{2}[f(x + t) + f(x - t) + 2t],$$

and so

$$u\left(\frac{3}{4}, \frac{7}{2}\right) = \frac{1}{2}\left[f\left(\frac{17}{4}\right) + f\left(-\frac{11}{4}\right) + 7\right].$$

Now $4 < \dfrac{17}{4} < 5$ and $-3 < -\dfrac{11}{4} < -2$;

therefore

$$f\left(\frac{17}{4}\right) = \left(\frac{1}{4}\right)^2\left(\frac{3}{4}\right)^2$$

and

$$f\left(-\frac{11}{4}\right) = \left(\frac{1}{4}\right)^2\left(\frac{3}{4}\right)^2,$$

using the formula for extending $\bar{f}$. This produces

$$u\left(\frac{3}{4}, \frac{7}{2}\right) = \frac{1}{2}\left[2\left(\frac{1}{4}\right)^2\left(\frac{3}{4}\right)^2 + 7\right]$$

$$= \frac{905}{256}.$$

Note that, for fixed x and increasing t, the terms $f(x + t)$ and $f(x - t)$ are bounded in magnitude but the integral of g increases indefinitely. This looks unusual but is due to the facts (if we are considering a vibrating string, say) that both ends of the string are free, and the whole string is given an initial velocity of unity in the positive u-direction

$$\frac{\partial u}{\partial t}(x, 0) = g(x) = 1.$$

The whole string will continue to move with this transverse velocity since neither end is constrained transversely. Superimposed on this constant velocity is the wave motion due to the initial displacement

$$u(x, 0) = f(x).$$

SAQ 12

W: page 24, Exercise 1

(Solution on p. 38.)

SAQ 13

Find u for $x \in [0, 1]$, $t \geqslant 0$ such that

$$\frac{\partial^2 u}{\partial t^2} - c^2\frac{\partial^2 u}{\partial x^2} = 0 \qquad (0 < x < 1, \quad t > 0),$$

$$u|_{t=0} = \sin \tfrac{1}{2}\pi x \qquad (0 \leqslant x \leqslant 1),$$

$$\left.\frac{\partial u}{\partial t}\right|_{t=0} = \sin \tfrac{3}{2}\pi x \qquad (0 \leqslant x \leqslant 1),$$

$$u|_{x=0} = 0 \qquad (t \geqslant 0),$$

$$\left.\frac{\partial u}{\partial x}\right|_{x=1} = 0 \qquad (t \geqslant 0).$$

(Solution on p. 38.)

SAQ 14

In a vibrating string problem for a string of length l, there is a free boundary condition at the end $x = 0$ and a fixed boundary condition at the end $x = l$. When the initial condition functions

$$f : x \longmapsto u|_{t=0} \qquad (x \in [0, l])$$

and

$$g : x \longmapsto \left.\frac{\partial u}{\partial t}\right|_{t=0} \qquad (x \in [0, l])$$

are extended to help solve the problem, what is the period of the extended functions?

(Solution on p. 38.)

1.3 SUMMARY

The *wave equation*

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0$$

has been shown to describe approximately:

(a) transverse displacements of a stretched string;

(b) longitudinal vibrations in an elastic bar; and

(c) one-dimensional sound waves in a gas.

The *homogeneous* wave equation on $[0, l] \times [0, \infty)$ has been solved over the whole future domain ($t \in [0, \infty)$) in terms of the *initial conditions* for $x \in [0, l]$ and *fixed* or *free boundary conditions*. The general solution is

$$\tfrac{1}{2}[f(x + ct) + f(x - ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(\bar{x})d\bar{x},$$

where

$$f: x \longmapsto u|_{t=0} \qquad (x \in [0, l]),$$

and

$$g: x \longmapsto \left.\frac{\partial u}{\partial t}\right|_{t=0} \qquad (x \in [0, l]),$$

and the domains of f and g are extended to the whole real line so that they are *odd* about an end point with a fixed boundary condition and *even* about an end point with a free boundary condition.

1.4 FURTHER SELF-ASSESSMENT QUESTIONS

SAQ 15

Which of the following statements, about the solution of the homogeneous wave equation for a finite string, is/are true?

A At each point the solution depends on the initial conditions and the boundary conditions.

B At each point the solution depends only on the boundary conditions.

C On a certain domain the solution depends only on the initial conditions, but elsewhere it depends on both the initial and boundary conditions.

D On a certain domain the solution depends only on the boundary conditions, but elsewhere it depends on both the initial and boundary conditions.

E At each point the solution depends only on the initial conditions.

(Solution on p. 39.)

SAQ 16

In solving the homogeneous wave equation in the domain $0 \leqslant x \leqslant 1$, $t \geqslant 0$ with initial conditions

$$f(x) = x^3 - 3x \qquad (0 \leqslant x \leqslant 1),$$

$$g(x) = \sin \pi x \qquad (0 \leqslant x \leqslant 1),$$

and boundary conditions

$$u|_{x=0} = 0 \qquad (t \geqslant 0),$$

$$\left.\frac{\partial u}{\partial x}\right|_{x=1} = 0 \qquad (t \geqslant 0),$$

find the appropriate extensions of the functions f and g.

(Solution on p. 39.)

1.5 SOLUTIONS TO SELF-ASSESSMENT QUESTIONS

Solution to SAQ 1

We know that

$$\tan \psi = y',$$

from the definition of the tangent. Differentiation (using the composite function rule) leads to

$$\sec^2\psi\,\psi' = y''.$$

We obtain s' by using the approximate expression

$$\frac{\Delta s}{\Delta x} = \sec \psi(x).$$

Proceeding to the limit as Δx becomes small, we obtain

$$s' = \sec \psi.$$

Therefore

$$\begin{aligned} K &= \frac{\psi'}{s'} \\ &= \frac{y'' \cos^2\psi}{\sec \psi} \\ &= \frac{y''}{(1 + \tan^2\psi)^{3/2}} \\ &= \frac{y''}{(1 + y'^2)^{3/2}}. \end{aligned}$$

Therefore

$$K = \frac{y''}{[1 + (y')^2]^{3/2}}.$$

Solution to SAQ 2

If we apply Newton's Second Law of Motion to that portion of the rod between the planes $x = x_1$ and $x = x_2$ for any $x_1 < x_2$, we obtain

$$E\frac{\partial u}{\partial x}(x_2, t) + K\left[\frac{\partial u}{\partial x}(x_2, t)\right]^2 - E\frac{\partial u}{\partial x}(x_1, t) - K\left[\frac{\partial u}{\partial x}(x_1, t)\right]^2$$
$$= \int_{x_1}^{x_2} \rho(x)\left[\frac{\partial^2 u}{\partial t^2}(x, t) - F(x, t)\right]dx \qquad (t \geqslant 0).$$

Partial differentiation of this equation with respect to x_2 yields

$$E\frac{\partial^2 u}{\partial x^2} + 2K\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x^2} - \rho\frac{\partial^2 u}{\partial t^2} = -\rho F.$$

This is a non-linear equation, owing to the second term on the left-hand side.

Solution to SAQ 3

Since the table top is incompressible the ends of the bar will always be in the positions they occupied with the bar in equilibrium; thus the displacement will always be zero at these points. The boundary conditions on the displacement function u are then

$$u(0, t) = u(l, t) = 0 \qquad (t \geqslant 0).$$

Solution to SAQ 4

The ends of the bar are free, so that the tension at each end is zero:

$$T(0, t) = T(l, t) = 0 \qquad (t \geqslant 0).$$

Since

$$T = E\frac{\partial u}{\partial x},$$

we see that

$$\frac{\partial u}{\partial x}(0, t) = \frac{\partial u}{\partial x}(l, t) = 0 \qquad (t \geqslant 0)$$

are the appropriate boundary conditions.

Solution to SAQ 5

We use the equations on *W: pages 7 and 8*:

$$\frac{\partial^2 u}{\partial t^2} - c^2\frac{\partial^2 u}{\partial x^2} = \frac{\partial F}{\partial t}$$

and

$$\frac{\partial^2 \rho}{\partial t^2} - c^2\frac{\partial^2 \rho}{\partial x^2} = -\rho_0\frac{\partial F}{\partial x},$$

where $c^2 = p'(\rho_0)$. In our problem

$$p = \alpha\rho^{\gamma}.$$

Therefore

$$\frac{dp}{d\rho} = \gamma\alpha\rho^{\gamma-1}.$$

Therefore

$$\begin{aligned} c^2 &= p'(\rho_0) \\ &= \frac{\gamma p_0}{\rho_0}, \end{aligned}$$

where

$$p_0 = \alpha\rho_0^{\gamma}.$$

Also

$$F(x, t) = F_0 \sin ax \cos bt;$$

therefore

$$\frac{\partial F}{\partial t} = -bF_0 \sin ax \sin bt$$

and

$$-\rho_0\frac{\partial F}{\partial x} = -a\rho_0F_0 \cos ax \cos bt.$$

The required equations therefore are:

$$\frac{\partial^2 u}{\partial t^2} - \left(\frac{\gamma p_0}{\rho_0}\right)\frac{\partial^2 u}{\partial x^2} = -bF_0 \sin ax \sin bt$$

and

$$\frac{\partial^2 \rho}{\partial t^2} - \left(\frac{\gamma p_0}{\rho_0}\right)\frac{\partial^2 \rho}{\partial x^2} = -a\rho_0F_0 \cos ax \cos bt.$$

Solution to SAQ 6

The gas acceleration $a(x, t)$ is given by

$$a(x, t) = u(x, t)\frac{\partial u}{\partial x}(x, t) + \frac{\partial u}{\partial t}(x, t).$$

If $u(x, t)$ is independent of t then

$$\frac{\partial u}{\partial t}(x, t) = 0;$$

so that

$$a(x, t) = u(x, t)\frac{\partial u}{\partial x}(x, t).$$

The acceleration is independent of time but is zero only if u is a constant function.

An element of gas moves through space and can change its velocity in doing so even though the velocity at each point in space may stay constant. For example, consider gas flow in a cylinder in which the density varies along the axis but does not change with time.

Solution to SAQ 7

Using Equations (1.13) on *W: page 7* with $F = 0$, putting time derivatives equal to zero and using a prime (′) for $\partial/\partial x$ we obtain

$$\rho u' + u\rho' = 0$$

and

$$\rho u u' + p' = 0.$$

The first of these equations can be written as

$$(\rho u)' = 0,$$

which gives

$$\rho u = K,$$

where K is a constant.

Substituting this result into the second equation, we obtain

$$K u' + p' = 0.$$

Integration yields

$$K u + p = L,$$

where L is a constant, i.e.

$$p = -K u + L.$$

Solution to SAQ 8

We have

$$\frac{\partial^2 u}{\partial t^2} - c^2\frac{\partial^2 u}{\partial x^2} = 0 \qquad (0 < x < 1, \quad t > 0),$$

$$u|_{t=0} = f(x) = \sin \pi x \quad (0 \leqslant x \leqslant 1),$$

$$\left.\frac{\partial u}{\partial t}\right|_{t=0} = g(x) = 0 \qquad (0 \leqslant x \leqslant 1),$$

$$u|_{x=0} = u|_{x=1} = 0 \qquad (t \geqslant 0).$$

We can extend f and g as odd functions about $x = 0$ and $x = 1$ with domain the whole real line by defining

$$f(x) = \sin \pi x \qquad (x \in R)$$

and

$$g(x) = 0 \qquad (x \in R).$$

This satisfies the boundary conditions. Using the general solution

$$u = \tfrac{1}{2}[f(x + ct) + f(x - ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(\bar{x})d\bar{x}$$

we obtain

$$\begin{aligned} u &= \tfrac{1}{2}[\sin \pi(x + ct) + \sin \pi(x - ct)] \\ &= \sin \pi x \cos \pi ct. \end{aligned}$$

(We have used the trigonometric identity

$$\sin(A + B) + \sin(A - B) = 2 \sin A \cos B.)$$

The solution to this problem is an example of a *standing wave*. The outline of the string at any time is a half-period sine wave and all particles of the string oscillate in phase with the frequency $\frac{1}{2}c$, but with different amplitudes. Roughly speaking, it is the two-dimensional side view you would get when watching a skipping rope being swung round in circles between two children, each holding an end, moving their hands just sufficiently to keep the rope in motion.

Solution to SAQ 9

We use the natural extensions for f and g, which are odd about 0 and l, for the fixed boundary conditions:

$$f(x) = a \sin \frac{\pi x}{l} \qquad (x \in R),$$

and

$$g(x) = -b \sin \frac{\pi x}{l} \qquad (x \in R).$$

The general solution

$$u = \tfrac{1}{2}[f(x + ct) + f(x - ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(\bar{x})d\bar{x}$$

produces

$$\begin{aligned} u &= \frac{a}{2}\left[\sin \frac{\pi(x+ct)}{l} + \sin \frac{\pi(x-ct)}{l}\right] - \frac{b}{2c}\int_{x-ct}^{x+ct} \sin \frac{\pi \bar{x}}{l}\, d\bar{x} \\ &= a \sin \frac{\pi x}{l} \cos \frac{\pi ct}{l} - \frac{lb}{\pi c} \sin \frac{\pi x}{l} \sin \frac{\pi ct}{l} \\ &= \sin \frac{\pi x}{l}\left[a \cos \frac{\pi ct}{l} - \frac{lb}{\pi c} \sin \frac{\pi ct}{l}\right]. \end{aligned}$$

(We have used the trigonometric identities

$$\sin(A + B) + \sin(A - B) = 2 \sin A \cos B,$$

$$\cos(A - B) - \cos(A + B) = 2 \sin A \sin B.)$$

Whenever the time dependent factor is zero, that is, for values of t satisfying

$$\tan \frac{\pi ct}{l} = \frac{\pi ac}{lb},$$

u is zero for all $x \in [0, l]$. Therefore

$$u|_{t=t_n} = 0 \quad x \in [0, l]$$

at time t_n given by

$$t_n = \frac{l}{\pi c}\left[\tan^{-1}\frac{\pi ac}{lb}\right] + \frac{nl}{c} \qquad (n = 0, 1, 2, \ldots).$$

The time dependent factor in u can be expressed (after a considerable amount of manipulation) as

$$a\cos\frac{\pi ct}{l} - \frac{lb}{\pi c}\sin\frac{\pi ct}{l} = -\sqrt{a^2 + \left(\frac{lb}{\pi c}\right)^2}\sin\left(\frac{\pi ct}{l} - \phi\right)$$

where $\tan\phi = \dfrac{\pi ac}{lb}$. Thus u has the form of a standing wave (see solution to SAQ 8) which occupies the equilibrium position at times $t = t_n$ $(n = 0, 1, 2, \ldots)$.

Solution to SAQ 10

Let $y(x, t)$ be the displacement at time t of the element of string at x. The acceleration of the bead is

$$\frac{\partial^2 y}{\partial t^2}(0, t),$$

its mass is m and the y-component of the force acting on it is $T\sin\psi(t)$ where $\psi(t)$ is the angle at time t between the tangent to the string at $x = 0$ and the x-axis. Newton's Second Law of Motion gives

$$T\sin\psi(t) = m\frac{\partial^2 y}{\partial t^2}(0, t).$$

Now

$$\begin{aligned}\sin\psi &= \cos\psi\tan\psi\\ &= \frac{\tan\psi}{\sec\psi}\\ &= \frac{\tan\psi}{\sqrt{1+\tan^2\psi}},\end{aligned}$$

and

$$\tan\psi(t) = \frac{\partial y}{\partial x}(0, t).$$

Thus

$$\frac{T\dfrac{\partial y}{\partial x}(0, t)}{\sqrt{1 + \left[\dfrac{\partial y}{\partial x}(0, t)\right]^2}} = m\frac{\partial^2 y}{\partial t^2}(0, t).$$

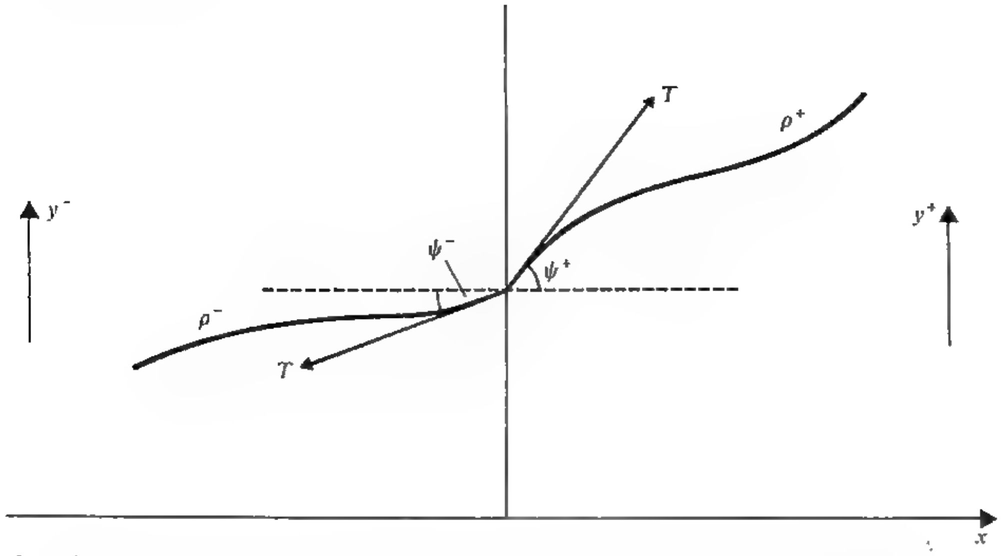
Since the transverse vibrations are small we may assume that

$$\frac{\partial y}{\partial x} \ll 1;$$

we obtain, in this approximation, the linear boundary condition

$$T\frac{\partial y}{\partial x}(0, t) = m\frac{\partial^2 y}{\partial t^2}(0, t).$$

Solution to SAQ 11



Clearly the string is continuous at $x = 0$. Therefore one boundary condition is

$$y^-(0, t) = y^+(0, t) \qquad (t \geqslant 0).$$

This, of course, implies that

$$\frac{\partial y^-}{\partial t}(0, t) = \frac{\partial y^+}{\partial t}(0, t) \qquad (t \geqslant 0)$$

and

$$\frac{\partial^2 y^-}{\partial t^2}(0, t) = \frac{\partial^2 y^+}{\partial t^2}(0, t) \quad (t \geqslant 0).$$

Let the tangents to the string at $x = 0$ make angles ψ^- and ψ^+ with the x-axis on the negative and positive sides of $x = 0$, respectively. Consider a small element of string near the junction between $x = -\frac{1}{2}\Delta x$ and $x = \frac{1}{2}\Delta x$. It has mass

$$\tfrac{1}{2}(\rho^- \sec\psi^- + \rho^+ \sec\psi^+)\Delta x$$

and acceleration

$$\frac{\partial^2 y^-}{\partial t^2}(0, t)\left(= \frac{\partial^2 y^+}{\partial t^2}(0, t)\right).$$

The y-component of the force acting on the element is

$$T(\sin\psi^+ - \sin\psi^-).$$

Newton's Second Law of Motion now gives us

$$T(\sin\psi^+ - \sin\psi^-) = \tfrac{1}{2}(\rho^- \sec\psi^- + \rho^+ \sec\psi^+)\Delta x \frac{\partial^2 y^-}{\partial t^2}(0, t) \qquad (t \geqslant 0).$$

Proceeding to the limit as Δx approaches zero, we obtain

$$\sin\psi^+ = \sin\psi^-.$$

i.e.

$$\tan\psi^+ = \tan\psi^-.$$

But

$$\tan\psi = \frac{\partial y}{\partial x}$$

so that

$$\frac{\partial y^-}{\partial x}(0, t) = \frac{\partial y^+}{\partial x}(0, t) \qquad (t \geqslant 0).$$

This is the second boundary condition at the junction.

Solution to SAQ 12

The equation is

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = 0 \qquad (0 < x < \tfrac{1}{2}\pi,\ t > 0).$$

We have the fixed boundary condition

$$u(0, t) = 0 \qquad (t \geqslant 0),$$

and the free boundary condition

$$\frac{\partial u}{\partial x}(\tfrac{1}{2}\pi, t) = 0 \qquad (t \geqslant 0).$$

The functions

$$f : x \longmapsto \sin x \qquad (x \in R)$$

and

$$g : x \longmapsto 0 \qquad (x \in R)$$

are odd about $x = 0$ and even about $x = \frac{1}{2}\pi$, as required by the boundary conditions. Using the general formula for the solution

$$u(x, t) = \tfrac{1}{2}[f(x + ct) + f(x - ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(\bar{x})d\bar{x},$$

we obtain, since $c = 1$,

$$\begin{aligned} u(x, t) &= \tfrac{1}{2}[\sin(x + t) + \sin(x - t)] \\ &= \sin x \cos t \qquad (t \geqslant 0,\ 0 \leqslant x \leqslant \tfrac{1}{2}\pi). \end{aligned}$$

Solution to SAQ 13

The functions

$$f : x \longmapsto \sin \tfrac{1}{2}\pi x \qquad (0 \leqslant x \leqslant 1)$$

and

$$g : x \longmapsto \sin \tfrac{3}{2}\pi x \qquad (0 \leqslant x \leqslant 1)$$

are odd about $x = 0$ and even about $x = 1$ and are therefore appropriate extensions of the initial conditions for the given boundary conditions. Using the general formula for the solution we obtain

$$\begin{aligned} u &= \tfrac{1}{2}[f(x + ct) + f(x - ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(\bar{x})d\bar{x} \\ &= \sin \tfrac{1}{2}\pi x \cos \tfrac{1}{2}\pi ct + \frac{2}{3\pi c}\sin \tfrac{3}{2}\pi x \sin \tfrac{3}{2}\pi ct \qquad (0 \leqslant x \leqslant 1,\ t \geqslant 0). \end{aligned}$$

Solution to SAQ 14

There is a free boundary condition at $x = 0$ so that we require

$$f(x) = f(-x) \quad \text{and} \quad g(x) = g(-x).$$

There is a fixed boundary condition at $x = l$ so that we require

$$f(l + x) = -f(l - x) \quad \text{and} \quad g(l + x) = -g(l - x).$$

Putting $-y = l + x$ in the second equation for f we obtain

$$f(-y) = -f(2l + y).$$

But

$$f(y) = f(-y),$$

and therefore

$$f(y) = -f(2l + y).$$

Replacing y by $(2l + y)$ in the last equation, we obtain

$$f(2l + y) = -f(2l + (2l + y)) = -f(4l + y).$$

The last two results give us

$$f(y) = f(4l + y).$$

Similarly,

$$g(y) = g(4l + y).$$

Thus, when we have a free boundary condition at one end and a fixed boundary condition at the other, the period of the extended initial condition functions is $4l$.

When both boundary conditions are free or both fixed the period is $2l$.

Solution to SAQ 15

The only true statement is **C**.

Solution to SAQ 16

We can extend f and g as even functions about $x = 1$ and odd functions about $x = 0$ by using the definitions

$$f(x) = \begin{cases} x^3 - 3x & (-1 \leqslant x \leqslant 1) \\ (2 - x)^3 - 3(2 - x) & (1 \leqslant x \leqslant 2) \\ -(2 + x)^3 + 3(2 + x) & (-2 \leqslant x \leqslant -1) \\ f(x - 4n) & (4n - 2 \leqslant x \leqslant 4n + 2,\ n \text{ an integer}). \end{cases}$$

and

$$g(x) = \begin{cases} |\sin \pi x| & (0 \leqslant x \leqslant 2) \\ -|\sin \pi x| & (-2 \leqslant x \leqslant 0) \\ g(x - 4n) & (4n - 2 \leqslant x \leqslant 4n + 2,\ n \text{ an integer}). \end{cases}$$

Unit 2 Classification and Characteristics

Contents

Set Books

G. D. Smith, *Numerical Solution of Partial Differential Equations* (Oxford, 1971).
H. F. Weinberger, *A First Course in Partial Differential Equations* (Blaisdell, 1965).

It is essential to have these books; the course is based on them and will not make sense without them. They are referred to in the text as *S* and *W* respectively.

Unit 2 is based on *W*: Chapter II, Sections 6 to 9.

Conventions

Before working through this text make sure you have read *A Guide to the Course: Partial Differential Equations of Applied Mathematics.* References to Open University courses in mathematics take the form:

Unit M100 13, Integration II for the Mathematics Foundation Course,
Unit M201 23, The Wave Equation for the Linear Mathematics Course.

2.0 INTRODUCTION

In this text we are concerned with linear partial differential equations which have linear subsidiary conditions, and, in particular, we examine second-order equations in two variables.

The text divides into three main parts. Section 2.1 concerns the linearity properties of equations and their subsidiary conditions, and introduces the *principle of superposition* for nonhomogenous problems. In the second section, second-order equations are classified into three types and *characteristic curves* are discussed. Finally, the *numerical method of characteristics* is used to obtain numerical solutions of hyperbolic equations.

2.1 LINEARITY AND SUPERPOSITION

2.1.1 The Linear Partial Differential Equation

In *Unit 1, The Wave Equation* we obtained the solution to the wave equation

$$\frac{\partial^2 u}{\partial t^2} - c^2\frac{\partial^2 u}{\partial x^2} = 0$$

in the form

$$u = p(\xi) + q(\eta),$$

where p and q are any twice differentiable functions of one variable and $\xi = x + ct$, $\eta = x - ct$ (see *W: page 9*).

In this section we are going to generalize this form and demonstrate that it represents solutions of a wider class of linear partial differential equations.

Suppose that a function $(x, t) \longmapsto u$ is given by

$$u = p(\xi) + q(\eta) + \zeta,$$

where p and q are arbitrary functions of one variable, and ξ, η and ζ are variables dependent on x and t, but not necessarily according to

$$\xi = x + ct, \eta = x - ct, \zeta = 0$$

as in *Unit 1*.

The only constraint we impose on the prescribed functions

$$(x, t) \longmapsto \xi,$$

$$(x, t) \longmapsto \eta,$$

$$(x, t) \longmapsto \zeta,$$

is that they be twice differentiable.

We partially differentiate the expression for u, with respect to x to obtain $\partial u/\partial x$, and with respect to t to obtain $\partial u/\partial t$. By the chain rule for functions of several variables,

$$\frac{\partial u}{\partial x} = p'(\xi)\frac{\partial \xi}{\partial x} + q'(\eta)\frac{\partial \eta}{\partial x} + \frac{\partial \zeta}{\partial x}$$

and

$$\frac{\partial u}{\partial t} = p'(\xi)\frac{\partial \xi}{\partial t} + q'(\eta)\frac{\partial \eta}{\partial t} + \frac{\partial \zeta}{\partial t},$$

where p' and q' are the derived functions of p and q respectively. The second-order derivatives of u are given by

$$\frac{\partial^2 u}{\partial x^2} = p''(\xi)\left(\frac{\partial \xi}{\partial x}\right)^2 + p'(\xi)\frac{\partial^2 \xi}{\partial x^2} + q''(\eta)\left(\frac{\partial \eta}{\partial x}\right)^2 + q'(\eta)\frac{\partial^2 \eta}{\partial x^2} + \frac{\partial^2 \zeta}{\partial x^2},$$

$$\frac{\partial^2 u}{\partial x\partial t} = p''(\xi)\frac{\partial \xi}{\partial x}\frac{\partial \xi}{\partial t} + p'(\xi)\frac{\partial^2 \xi}{\partial x\partial t} + q''(\eta)\frac{\partial \eta}{\partial x}\frac{\partial \eta}{\partial t} + q'(\eta)\frac{\partial^2 \eta}{\partial x\partial t} + \frac{\partial^2 \zeta}{\partial x\partial t}$$

and

$$\frac{\partial^2 u}{\partial t^2} = p''(\xi)\left(\frac{\partial \xi}{\partial t}\right)^2 + p'(\xi)\frac{\partial^2 \xi}{\partial t^2} + q''(\eta)\left(\frac{\partial \eta}{\partial t}\right)^2 + q'(\eta)\frac{\partial^2 \eta}{\partial t^2} + \frac{\partial^2 \zeta}{\partial t^2}.$$

We now have a set of five equations involving the four quantities $p''(\xi)$, $p'(\xi)$, $q''(\eta)$ and $q'(\eta)$. Moreover, the equations are linear in these quantities, and may therefore be written in the form

$$\begin{bmatrix} 0 & \dfrac{\partial \xi}{\partial x} & 0 & \dfrac{\partial \eta}{\partial x} & \dfrac{\partial \zeta}{\partial x} - \dfrac{\partial u}{\partial x} \\ 0 & \dfrac{\partial \xi}{\partial t} & 0 & \dfrac{\partial \eta}{\partial t} & \dfrac{\partial \zeta}{\partial t} - \dfrac{\partial u}{\partial t} \\ \left(\dfrac{\partial \xi}{\partial x}\right)^2 & \dfrac{\partial^2 \xi}{\partial x^2} & \left(\dfrac{\partial \eta}{\partial x}\right)^2 & \dfrac{\partial^2 \eta}{\partial x^2} & \dfrac{\partial^2 \zeta}{\partial x^2} - \dfrac{\partial^2 u}{\partial x^2} \\ \dfrac{\partial \xi}{\partial x}\dfrac{\partial \xi}{\partial t} & \dfrac{\partial^2 \xi}{\partial x \partial t} & \dfrac{\partial \eta}{\partial x}\dfrac{\partial \eta}{\partial t} & \dfrac{\partial^2 \eta}{\partial x \partial t} & \dfrac{\partial^2 \zeta}{\partial x \partial t} - \dfrac{\partial^2 u}{\partial x \partial t} \\ \left(\dfrac{\partial \xi}{\partial t}\right)^2 & \dfrac{\partial^2 \xi}{\partial t^2} & \left(\dfrac{\partial \eta}{\partial t}\right)^2 & \dfrac{\partial^2 \eta}{\partial t^2} & \dfrac{\partial^2 \zeta}{\partial t^2} - \dfrac{\partial^2 u}{\partial t^2} \end{bmatrix} \begin{bmatrix} p''(\xi) \\ p'(\xi) \\ q''(\eta) \\ q'(\eta) \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Since the column vector $(p''(\xi), p'(\xi), q''(\eta), q'(\eta), 1)$* is not the zero vector, the matrix in the equation must be singular, and so its determinant is equal to 0. Now, ξ, η and ζ are known variables, and derivatives of u only occur in the right-hand column of the matrix. As a result, when the determinant is expanded and set equal to zero, we obtain

$$A(x,t)\frac{\partial^2 u}{\partial x^2} + B(x,t)\frac{\partial^2 u}{\partial x \partial t} + C(x,t)\frac{\partial^2 u}{\partial t^2} + D(x,t)\frac{\partial u}{\partial x} + E(x,t)\frac{\partial u}{\partial t} + F(x,t) = 0,$$

where A, B, C, D, E and F are known functions of two variables. Thus

$$u = p(\xi) + q(\eta) + \zeta$$

is the solution of a second-order *linear* partial differential equation and we can always obtain such an equation from an expression of this kind.

It is important to realize at this point that we are not proposing a solution of the above form for *all* linear partial differential equations. We shall see in Section 2.2 that the equations of a particular, well-defined class have solutions of the above form.

You should check that if ξ, η and ζ are given by

$$\xi = x + ct,$$

$$\eta = x - ct,$$

$$\zeta = 0,$$

then we obtain the equation

$$\frac{\partial^2 u}{\partial t^2} - c^2\frac{\partial^2 u}{\partial x^2} = 0,$$

as expected.

SAQ 1

If $u = p(x^2 - t) + q(x^2 + t)$, where p and q are arbitrary functions of one variable, prove that

$$\frac{\partial^2 u}{\partial x^2} - \frac{1}{x}\frac{\partial u}{\partial x} - 4x^2\frac{\partial^2 u}{\partial t^2} = 0.$$

(Solution on p. 28.)

* For typographical convenience an n-tuple may denote a $n \times 1$ column matrix.

SAQ 2

Show that, if p and q are arbitrary functions of a single complex variable, then

$$u = p(x + \alpha t + i\beta y) + q(x - \alpha t - i\beta y)$$

is a solution of

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial t^2},$$

provided that $\alpha^2 + \beta^2 = 1$.

(Solution on p. 28.)

SAQ 3

Prove that

$$u = [p(\xi) + q(\eta)]^{1/2},$$

where p and q are arbitrary twice differentiable functions of a single variable and where ξ, η are variables dependent on x and t, satisfies a second-order (nonlinear) partial differential equation.

(Solution on p. 28.)

We shall now proceed to consider the linearity aspect in more detail. Before reading the next passage, however, you may find it helpful to review *Unit M201 23, The Wave Equation*, Section 23.1.2.

READ W: Section 6, page 29 to page 32, line 21 (the end of the example).

You may ignore the references in the two examples to Section 5 of ***W*** which is not included in this course.

Notes

(i) *W: page 29, line −5*
Not only do most physical problems involve nonlinear operators—the resulting equations are usually either very difficult or impossible to solve analytically. However, where they can be approximated by linear operators, solutions may well be obtained which give a good idea of the ways in which to tackle the non-linear problem.

(ii) *W: page 31, line 6*
The solution set of every homogeneous linear partial differential equation is an infinite-dimensional vector space (*Unit M201 23*). Similarly the set of functions satisfying the homogeneous linear subsidiary condition

$$L_i[u] = 0$$

is an infinite-dimensional vector space, in general. The general solution of the homogenous linear problem is the intersection of all these spaces, which may be finite-dimensional.

(iii) *W: page 31*, **Example**
The problem may be interpreted physically in terms of a string whose ends are fixed at $x = 0$ and $x = 1$, which is subject to a transverse force which varies along its length as $\sin \pi x$. (Note that the ends are subject to no body force.) At time $t = 0$ the string is released from a stationary position along the x-axis.

The steady-state (or equilibrium) solution of the partial differential equation with the boundary conditions (but not the initial conditions) is given by $v(x, t)$ subject to

$$\frac{\partial^2 v}{\partial t^2} = 0.$$

The equation now reduces to an ordinary differential equation whose solution (subject to the boundary conditions) is

$$v(x, t) = \frac{1}{\pi^2} \sin \pi x.$$

If we put $w = u - v$, the problem for w may be interpreted as a string with no external force released from an initial displacement distribution

$$-\frac{1}{\pi^2} \sin \pi x.$$

This problem is the vibrating string problem with $c = 1$ and $l = 1$, and its solution is given by Equation (2.16) on *W: page 13*. When we superpose the steady displacement

$$v(x, t) = \frac{1}{\pi^2} \sin \pi x$$

we obtain the standing-wave solution

$$u(x, t) = \frac{1}{\pi^2} \sin \pi x(1 - \cos \pi t).$$

This is readily seen to be a standing wave by observing that $x = 0$ and $x = 1$ are points of zero displacement for all time. In other words, the wave has the form $A(t) \sin \pi x$, where the profile does not change its basic shape with time—just its amplitude varies.

SAQ 4

If

$$\begin{aligned} \frac{\partial^2 u}{\partial t^2}(x, t) - \frac{\partial^2 u}{\partial x^2}(x, t) &= 2 \cos x - x \sin x && 0 < x < \pi,\ t > 0, \\ u(x, 0) &= 0 && 0 \leqslant x \leqslant \pi, \\ \frac{\partial u}{\partial t}(x, 0) &= 0 && 0 \leqslant x \leqslant \pi, \\ u(0, t) &= 0 && t \geqslant 0, \\ u(\pi, t) &= 0 && t \geqslant 0, \end{aligned}$$

determine $u(\frac{1}{2}\pi, \frac{1}{2}\pi)$.

(Solution on p. 29.)

SAQ 5

Which of the following are linear operators?

A $\quad L: u \longmapsto \dfrac{\partial u}{\partial t} + u\left(\dfrac{\partial u}{\partial x} + 1\right)$

B $\quad L: u \longmapsto \left(\dfrac{\partial u}{\partial t}\right)^2 + \dfrac{\partial u}{\partial x} + \dfrac{\partial^2 u}{\partial x^2}$

C $\quad L: u \longmapsto \dfrac{\partial u}{\partial t} + e^x \dfrac{\partial^2 u}{\partial x^2}$

D $\quad L: u \longmapsto t\dfrac{\partial^2 u}{\partial t^2} + x^2 \dfrac{\partial u}{\partial x} + u$

E $\quad L: u \longmapsto \dfrac{\partial u}{\partial t} + \dfrac{\partial^2 u}{\partial x^2} + \dfrac{1}{x}\dfrac{\partial u}{\partial x} + \cos u$

(Solution on p. 30.)

SAQ 6

If

$$\frac{\partial^2 u}{\partial t^2}(x,t) - \frac{\partial^2 u}{\partial x^2}(x,t) = 2(1-x)\cos 2t \qquad 0 < x < 1,\ t > 0,$$

$$u(x,0) = 0 \qquad 0 \leqslant x \leqslant 1,$$

$$\frac{\partial u}{\partial t}(x,0) = 0 \qquad 0 \leqslant x \leqslant 1,$$

$$u(0,t) = \sin^2 t \qquad t \geqslant 0,$$

$$u(1,t) = 0 \qquad t \geqslant 0,$$

find $u(\frac{1}{2}, \frac{1}{6}\pi)$.

(Solution on p. 30.)

2.1.2 The Principle of Superposition

READ W: page 32, line 22 to page 35 (the end of Section 6), omitting the final sentence on page 32.

Notes

(i) *W: page 33, line −10 and footnote*
A series of functions converges uniformly if it is convergent in the uniform norm (*Unit M201 19, Least-Squares Approximation*). If a series of functions converges pointwise to g, a sufficient condition for term-by-term differentiation to be valid, i.e. for the differentiated series to converge to g′, is that the differentiated series converge uniformly. The footnote provides an alternative criterion for uniform convergence.

Do not worry if this is not clear at this stage—it will be discussed again in *Unit 6, Fourier Series.*

(ii) *W: pages 33 and 34*, **Example**
Do not confuse i here with $\sqrt{-1}$!

In this example $v^{(i)}$ satisfies the equations at the top of *W: page 33*, with the operators given by

$$L[u] \equiv \frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2},$$

$$L_1[u] \equiv x \longmapsto u(x,0),$$

$$L_2[u] \equiv x \longmapsto \frac{\partial u}{\partial t}(x,0),$$

$$L_3[u] \equiv x \longmapsto u(0,t),$$

$$L_4[u] \equiv x \longmapsto u(1,t).$$

and

$$L_1[v^{(i)}](x) = \sin i\pi x \qquad i \in Z^+.$$

The problem specifies (*W: page 34*)

$$L_1[u](x) = \sum_{i=1}^{\infty} i^{-4} \sin i\pi x$$

$$= \sum_{i=1}^{\infty} i^{-4} L_1[v^{(i)}](x).$$

Therefore,

$$u = \sum_{i=1}^{\infty} i^{-4} v^{(i)}$$

is a solution, since the series obtained by applying the operators L, L_1, L_2, L_3 and L_4 to u converge uniformly—a fact which we do not propose to prove here!

(iii) *W: page 34, line 21*

The uniqueness problem for the solution of the linear problem (6.1) on *W: page 30* depends only upon the linear operators $L, L_1, \ldots, L_k$ and is independent of the particular data $F, f_1, \ldots, f_k$. In other words, it depends upon the linear partial differential equation having specific types of linear boundary conditions, the actual functional forms of which do not affect the uniqueness of the solution if one exists. Uniqueness is determined without precise knowledge of the problem; it is sufficient to know that the corresponding homogeneous problem has a unique solution.

SAQ 7

Show that the problem

$$\begin{aligned}
\frac{\partial^2 u}{\partial t^2}(x,t) - c^2\frac{\partial^2 u}{\partial x^2}(x,t) &= F(x,t) && 0 < x < l, t > 0,\\
u(x,0) &= f(x) && 0 \leqslant x \leqslant l,\\
\frac{\partial u}{\partial t}(x,0) &= g(x) && 0 \leqslant x \leqslant l,\\
u(0,t) &= 0 && t \geqslant 0,\\
\frac{\partial u}{\partial x}(l,t) &= 0 && t \geqslant 0,
\end{aligned}$$

has, at most, one solution.

(Solution on p. 31.)

2.1.3 Uniqueness for the Vibrating String Problem

We have seen in *Unit 1* that the wave equation with constant coefficients can be solved, uniquely, for fixed and free boundary conditions. In terms of the initial condition functions for u and $\partial u/\partial t$ we found that

$$u(x,t) = \tfrac{1}{2}[f(x+ct) + f(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(\bar{x})d\bar{x}.$$

Clearly a change in f affects $u(x,t)$ only at $x - ct$ or $x + ct$. Thus the effect of an initial displacement is propagated at precisely the speed c. However $u(x,t)$ is affected by a change of g anywhere in the interval $[x - ct, x + ct]$. Thus the effect of a change in the initial velocity is propagated at all speeds up to c.

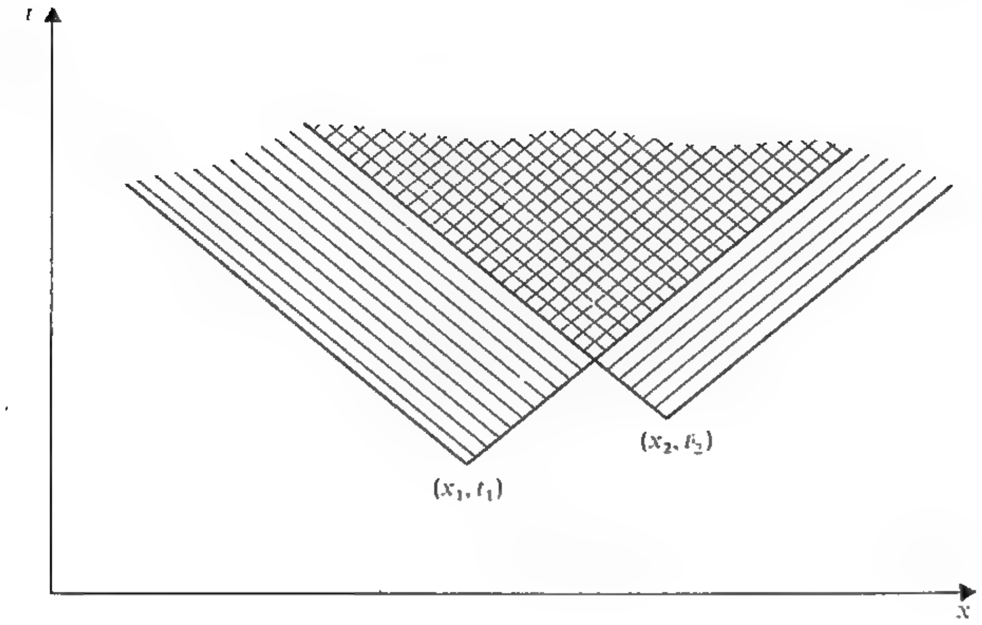
Now consider the point (x_1, t_1) and the two straight lines

$$x - ct = \text{constant}$$

and

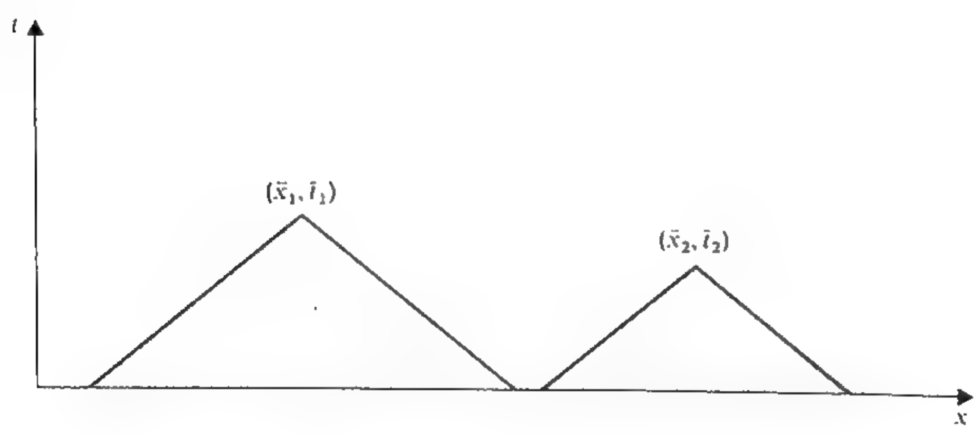
$$x + ct = \text{constant}$$

which pass through it. These lines represent propagation with the maximum speed c in the positive and negative x-directions, and are called **characteristics** of the wave equation. A pair of characteristics passes through each point (x_1, t_1) in the domain of the equation.



If a disturbance is concentrated at (x_1, t_1), it can influence only that sector of the half-plane $t \geqslant t_1$ formed by the two lines of slope $\pm 1/c$ passing through the point (x_1, t_1). Hence this sector is called the **domain of influence** of (x_1, t_1). Similarly we can form the domain of influence of the point (x_2, t_2). Note that all domains of influence intersect for $t \geqslant 0$.

The **domain of dependence** of the point $(\bar{x}, \bar{t})$ is the set of all points with $t \geqslant 0$ whose domains of influence include the point $(\bar{x}, \bar{t})$. Thus the domain of dependence of a point is the domain cut off from the upper half-plane by the two downward drawn lines of slope $\pm 1/c$. Clearly, since the domain of the wave equation does not include $t < 0$, it is possible to have non-intersecting domains of dependence. If the domains of dependence of, say, $(\bar{x}_1, \bar{t}_1)$ and $(\bar{x}_2, \bar{t}_2)$ do not intersect, the displacements at these points will not be coherent—they will be caused by initial positions and velocities which are independent of each other.



Because the pair of characteristics through a point bound its domain of influence, it turns out that if there is a discontinuity in a derivative of the solution across a given characteristic at some point, then this discontinuity propagates along the characteristic. We shall not prove this result, or even formulate it precisely, but we shall provide an illustration.

Example

Consider the problem

$$\frac{\partial^2 u}{\partial t^2} - c^2\frac{\partial^2 u}{\partial x^2} = 0 \qquad (x, t) \in R \times R^+ \text{ (the upper half-plane)},$$

$$u(x, 0) = f(x) \qquad x \in R,$$

$$\frac{\partial u}{\partial t}(x, 0) = 0 \qquad x \in R.$$

As explained on *W: page 10*, the solution to this problem is given by

$$u(x, t) = \tfrac{1}{2}[f(x + ct) + f(x - ct)]$$

at all points in the domain for which f is twice differentiable. Suppose now that

$$f: x \longmapsto \begin{cases} x^2 & x \geqslant 0 \\ -x^2 & x < 0. \end{cases}$$

Then

$$u: (x, t) \longmapsto \begin{cases} x^2 + c^2t^2 & 0 \leqslant t < x/c \\ 2ctx & t \geqslant |x/c| \\ -(x^2 + c^2t^2) & 0 \leqslant t < -x/c. \end{cases}$$

The functions u, $\partial u/\partial x$ and $\partial u/\partial t$ are continuous throughout the domain $R \times R^+$. However, the partial derived functions

$$\frac{\partial^2 u}{\partial x^2}, \frac{\partial^2 u}{\partial x \partial t} \quad \text{and} \quad \frac{\partial^2 u}{\partial t^2}$$

are not defined along the lines $x - ct = 0$ and $x + ct = 0$, their values undergoing jump discontinuities across these lines.

These lines are precisely the characteristics through $(0, 0)$, the point at which the initial condition function f does not have a second derivative. We say that the discontinuity (in f'') *propagates* along the characteristics.

We turn now to the more general vibrating string problem in which the coefficient $c^2(x)$ is no longer fixed along the string. The next reading passage develops the discussion of characteristics for this case in the context of proving that its solution is unique.

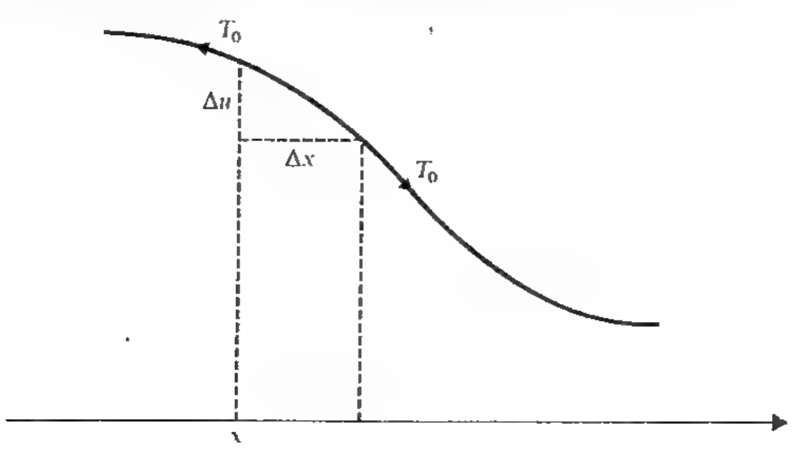
READ W: Section 7, pages 36 to 40.

Notes

(i) *W: page 36, lines 11 to −6*

This passage derives an integral form of the differential equation. Its physical interpretation is an application of the Law of Conservation of Energy to the string. This law is discussed in Section 8.1 of *Unit MST 282 8, Work and Energy II*. Briefly, it states that the change in the TOTAL ENERGY of a system is equal to the WORK done on it by all the forces.

We can obtain formulas for the *kinetic* and *potential* energies by studying the element of the string at x with undisturbed length Δx, at time t.



The *kinetic* energy of a particle is defined as

$$\tfrac{1}{2} \times \text{mass} \times (\text{velocity})^2.$$

For our element of string we obtain, approximately,

$$\tfrac{1}{2}\rho(x)\left[\frac{\partial u}{\partial t}(x, t)\right]^2 \Delta x.$$

The *potential* energy of a stretched string (i.e. the energy stored in the string) is equal to

$$\text{tension} \times \text{extension}.$$

The extension of our element of string is, approximately,

$$[(\Delta x)^2 + (\Delta u)^2]^{1/2} - \Delta x.$$

Since we are considering the situation at a given time t we find that the potential energy is, approximately,

$$T_0\left\{\left[1 + \left(\frac{\partial u}{\partial x}(x, t)\right)^2\right]^{1/2} - 1\right\}\Delta x \simeq \tfrac{1}{2}T_0\left(\frac{\partial u}{\partial x}(x, t)\right)^2 \Delta x,$$

on expanding the bracket by the binomial theorem and ignoring higher powers of $\partial u/\partial x$, which is assumed to be small, as in the derivation of the differential equation in *Unit 1*. The TOTAL ENERGY of the whole string at time t is then

$$\frac{1}{2}\int_0^l\left[\rho\left(\frac{\partial u}{\partial t}\right)^2 + T_0\left(\frac{\partial u}{\partial x}\right)^2\right]dx.$$

In addition, work is done in moving the ends of the string. The rate of work being done is

$$\text{force} \times \text{velocity}.$$

At each end of the string the rate of working is therefore the y-component of the tension multiplied by the transverse velocity.

The left-hand side of Equation (7.3) represents the total energy difference for the whole string at times $\bar{t}$ and 0. The Law of Conservation of Energy requires that this be equal to the work done on the system between these times by the external force.

F gives the external force per unit mass, so $F(x, t)\rho(x)\Delta x$ is the force on our element of string. Thus the rate of doing work on the element is

$$F(x, t)\rho(x)\frac{\partial u}{\partial t}\Delta x$$

and, for the whole string, this yields

$$\int_0^l F\rho\frac{\partial u}{\partial t}\,dx,$$

so that the WORK done at time $\bar{t}$ is

$$\int_0^{\bar{t}}\left[\int_0^l F\rho\frac{\partial u}{\partial t}\,dx\right]dt.$$

(ii) *W: page 37, line 14*

From

$$\frac{\partial v}{\partial t}(x, t_1) = 0 \qquad 0 < t_1 \leqslant \bar{t},$$

we deduce that $v(x, t_1) = v_0(x)$. Then the condition

$$v(x, 0) = 0 \quad \text{implies} \quad v_0(x) = 0,$$

since $t \longmapsto v(x, t)$ is continuous at 0.

(iii) *W: page 37, line −8*
Note that c depends upon x here, and we do not have characteristics which are straight lines as before. We therefore seek an alternative method of determining the domain of dependence of the point $(\bar{x}, \bar{t})$.

(iv) *W: page 37, line −1*
Using the formula for integration by substitution (backwards method), the last two terms of the left-hand side become

$$-\int_{\substack{t_2 \\ (x,t)\text{ on } C_2}}^{\bar{t}} T_0\frac{\partial v}{\partial x}(x,t)\frac{\partial v}{\partial t}(x,t)\,dt + \int_{\substack{t_1 \\ (x,t)\text{ on } C_1}}^{\bar{t}} T_0\frac{\partial v}{\partial x}(x,t)\frac{\partial v}{\partial t}(x,t)\,dt$$

$$= -\int_1^{\bar{x}} T_0\frac{\partial v}{\partial x}(x, Q_2(x))\frac{\partial v}{\partial t}(x, Q_2(x))Q_2'(x)\,dx$$

$$+\int_0^{\bar{x}} T_0\frac{\partial v}{\partial x}(x, Q_1(x))\frac{\partial v}{\partial t}(x, Q_1(x))Q_1'(x)\,dx$$

$$= \int_{\substack{0 \\ (x,t)\text{ on } C_1+C_2}}^{1} T_0\frac{\partial v}{\partial x}(x,t)\frac{\partial v}{\partial t}(x,t)\frac{dt}{dx}\,dx.$$

(v) *W: page 38, lines 12 to 19*
Note that

$$\int_{C_1+C_2} \tfrac{1}{2}\rho\left\{\left[\frac{\partial v}{\partial t} + c^2\frac{dt}{dx}\frac{\partial v}{\partial x}\right]^2 + c^4\left[\frac{1}{c^2} - \left(\frac{dt}{dx}\right)^2\right]\left(\frac{\partial v}{\partial x}\right)^2\right\}dx = 0$$

for all C_1 and C_2—in particular, when C_1, C_2 are given by

$$\left(\frac{dt}{dx}\right)^2 = \frac{1}{c^2(x)}.$$

In this case, the equation becomes

$$\int_{C_1} \tfrac{1}{2}\rho\left(\frac{\partial v}{\partial t} + c\frac{\partial v}{\partial x}\right)^2 dx + \int_{C_2} \tfrac{1}{2}\rho\left(\frac{\partial v}{\partial t} - c\frac{\partial v}{\partial x}\right)^2 dx = 0.$$

This is possible only if

$$\frac{\partial v}{\partial t} + c\frac{\partial v}{\partial x} = 0 \quad \text{on } C_1$$

and

$$\frac{\partial v}{\partial t} - c\frac{\partial v}{\partial x} = 0 \quad \text{on } C_2.$$

SAQ 8

W: page 40, Exercise 3

(Solution on p. 31.)

SAQ 9

W: page 40, Exercise 4

(Solution on p. 31.)

2.2 CLASSIFICATION OF SECOND-ORDER EQUATIONS

2.2.0 Introduction

We have seen that the solution of linear partial differential equations poses difficulties far greater than the solution of ordinary linear differential equations. However, for a particular problem—the wave equation—we were able to find a change of coordinates which allows us to obtain the solution in terms of initial and boundary conditions. In addition we were able to determine a uniqueness theorem for the "generalized" form of the wave equation in which the coefficient c is not constant. Our success in both cases exploits the existence of characteristics for the equations.

We now proceed to extend our approach to an arbitrary linear second-order partial differential equation in two variables.

Before continuing with this section, you may find it helpful to review Section 32.3 of *Unit M201 32, The Heat Conduction Equation*, which discusses how the plethora of second-order linear partial differential equations can be sorted into three categories, and the type of subsidiary conditions appropriate to each.

2.2.1 Second-Order Equations with Constant Coefficients

READ W: Section 8, pages 41 to 43.

Notes

(i) *W: page 41, line −3*
We specifically exclude $\alpha = \beta = 0$ and $\gamma = \delta = 0$, so that, if $A \neq 0$, the equations on *W: page 41, lines −7 and −6* imply that $\alpha \neq 0$ and $\gamma \neq 0$.

(ii) *W: page 42, line 3*
This means that ξ and η should be linearly independent.

(iii) *W: page 42, Equation (8.5)*
In obtaining this equation, we required $A \neq 0$. If, in addition, $C = 0$, Equations (8.4) become

$$\xi = 2Ax,$$

$$\eta = 2Ax - 2Bt,$$

or alternatively,

$$\xi = x,$$

$$\eta = x - \frac{B}{A}t,$$

which gives the same ratios β/α and δ/γ. The hyperbolic condition becomes $B^2 > 0$, which holds for all (real) B.

The case $A = 0$, $C \neq 0$ can be deduced immediately from the symmetry of Equation (8.1), and we can write

$$\xi = 2Ct,$$

$$\eta = 2Ct - 2Bx,$$

or

$$\xi = t,$$

$$\eta = x - \frac{C}{B}t.$$

(iv) *W: page 42, line −8*
We require

$$A\beta^2 + B\alpha\beta + C\alpha^2 = 0,$$

or

$$A\frac{\beta^2}{\alpha^2} + B\frac{\beta}{\alpha} + C = 0.$$

Since $B^2 = 4AC$,

$$A\frac{\beta^2}{\alpha^2} + B\frac{\beta}{\alpha} + \frac{B^2}{4A} = 0,$$

and

$$\left(\frac{\beta}{\alpha} + \frac{B}{2A}\right)^2 = 0.$$

The coefficient of $\dfrac{\partial^2 u}{\partial\xi\partial\eta}$ is now

$$\begin{aligned}2A\beta\delta + B(\alpha\delta + \beta\gamma) + \frac{B^2}{2A}\alpha\gamma &= \alpha\left\{\frac{2A\beta\delta}{\alpha} + B\left(\delta + \frac{\beta\gamma}{\alpha}\right) + \frac{B^2}{2A}\gamma\right\}\\ &= \alpha\left\{-B\delta + B\left(\delta - \frac{B}{2A}\gamma\right) + \frac{B^2}{2A}\gamma\right\}\\ &= 0.\end{aligned}$$

(v) *W: page 42, line − 1*
The *standard form* of a linear partial differential operator (or equation) is more usually called the *canonical form*.

(vi) *W: page 43, line 2*
$A = 0 \Rightarrow B = 0$, since $B^2 = 4AC$ in the parabolic case.

(vii) *W: page 43, Equations (8.8) and (8.9)*
Since we cannot make the coefficients of $\partial^2 u/\partial\xi^2$ or $\partial^2 u/\partial\eta^2$ vanish, we make the coefficient of $\partial^2 u/\partial\xi\partial\eta$ vanish instead:

$$2A\beta\delta + B(\alpha\delta + \beta\gamma) + 2C\alpha\gamma = 0$$

or

$$2A + B\left(\frac{\alpha}{\beta} + \frac{\gamma}{\delta}\right) + 2C\frac{\alpha}{\beta}\frac{\gamma}{\delta} = 0.$$

We choose

$$\frac{\gamma}{\delta} = 0 \text{ and } \frac{\alpha}{\beta} = -\frac{2A}{B}.$$

Thus (8.8) is the desired transformation, the denominator being chosen as $\sqrt{4AC - B^2}$ so that

$$L[u] = A\left[\frac{\partial^2 u}{\partial\xi^2} + \frac{\partial^2 u}{\partial\eta^2}\right].$$

(viii) *W: page 43, line 19*
Laplace's equation has no (real) characteristics along which discontinuities would propagate. The conclusion drawn by Weinberger must be taken on trust.

The general solution of Laplace's equation is not as straightforward as the others and will be studied in later units.

SAQ 10

W: page 44, Exercise 2

(Solution on p. 32.)

SAQ 11

W: page 43, Exercise 1

(Solution on p. 32.)

SAQ 12

Reduce the equation

$$\frac{\partial^2 u}{\partial t^2} + 4\frac{\partial^2 u}{\partial x \partial t} + 4\frac{\partial^2 u}{\partial x^2} = 0$$

to standard form, and hence solve it.

(Solution on p. 33.)

2.2.2 Characteristic Curves

Before asking you to read *W: Section 9* on general second-order operators, we summarize what we already know about characteristics. Firstly, we know that they define domains of influence and dependence: secondly, discontinuities propagate along them. In order to look more closely into their nature, let us use the second-order operator

$$L: u \longmapsto A(x,t)\frac{\partial^2 u}{\partial t^2} + B(x,t)\frac{\partial^2 u}{\partial x \partial t} + C(x,t)\frac{\partial^2 u}{\partial x^2} + F\left(x, t, u, \frac{\partial u}{\partial t}, \frac{\partial u}{\partial x}\right).$$

We wish to consider the problem of determining the solution of the equation $L[u] = 0$ which takes prescribed values on a given curve Γ in its domain. By a **curve** in R^2 we mean a subset of R^2 given by

$$\{(x,t): x = \varphi(s), t = \theta(s), s \in I\}$$

where I is an interval on the real line and φ, θ are functions $I \longrightarrow R$. We usually restrict ourselves to a **piecewise continuously differentiable curve**: that is, one for which φ and θ are piecewise continuously differentiable on I and there are only a finite number of points in I at which the derivatives of φ and θ vanish simultaneously.

We shall specify **Cauchy data** on Γ, i.e. x, t, u, $\partial u/\partial x$ and $\partial u/\partial t$ are given for $s \in I$. In practice we do not specify both $\partial u/\partial x$ and $\partial u/\partial t$ at any point, because u, $\partial u/\partial x$ and $\partial u/\partial t$ cannot all be assigned arbitrarily along the curve. To show this, we suppose u is known along the curve, so that du/ds may be determined. Applying the chain rule we obtain

$$\frac{du}{ds} = \frac{\partial u}{\partial x}\frac{dx}{ds} + \frac{\partial u}{\partial t}\frac{dt}{ds},$$

and we see that $\partial u/\partial x$ and $\partial u/\partial t$ are not independent since $dx/ds = \varphi'(s)$ and $dt/ds = \theta'(s)$ are determined by the defining equations of the curve. For example, in the Vibrating String Problem (Equations (7.1) on *W: page 36*) we specified $\partial u/\partial t$ along the initial curve $\{(x,t): 0 \leqslant x \leqslant l, t = 0\}$ but did not specify explicitly $\partial u/\partial x$.

Now, on the curve Γ, put $p = \partial u/\partial x$ and $q = \partial u/\partial t$, so that, by the chain rule,

$$\frac{dp}{ds} = \frac{\partial^2 u}{\partial x^2}\frac{dx}{ds} + \frac{\partial^2 u}{\partial t \partial x}\frac{dt}{ds}$$

$$\frac{dq}{ds} = \frac{\partial^2 u}{\partial x \partial t}\frac{dx}{ds} + \frac{\partial^2 u}{\partial t^2}\frac{dt}{ds}.$$

Combining these with the second-order partial differential equation $L[u] = 0$, we have the following system of equations for $\partial^2 u/\partial t^2$, $\partial^2 u/\partial x \partial t$, $\partial^2 u/\partial x^2$ evaluated *on the curve* Γ:

$$M \begin{bmatrix} \dfrac{\partial^2 u}{\partial t^2} \\[2ex] \dfrac{\partial^2 u}{\partial x \partial t} \\[2ex] \dfrac{\partial^2 u}{\partial x^2} \end{bmatrix} = \begin{bmatrix} \dfrac{dq}{ds} \\[2ex] \dfrac{dp}{ds} \\[2ex] -F \end{bmatrix},$$

where M is the matrix

$$\begin{bmatrix} \dfrac{dt}{ds} & \dfrac{dx}{ds} & 0 \\[2ex] 0 & \dfrac{dt}{ds} & \dfrac{dx}{ds} \\[2ex] A & B & C \end{bmatrix},$$

and A, B, C, F denote the respective values of these functions. Thus, if det $M \neq 0$ we can easily solve for the second derivatives $\partial^2 u/\partial t^2$, $\partial^2 u/\partial x \partial t$, $\partial^2 u/\partial x^2$ on the curve since x, t, u and the first derivatives are specified.

By successively differentiating these equations it may be shown that higher derivatives of u of all orders are uniquely determined at each point on Γ for which det $M \neq 0$. The values of the function u at neighbouring points can now be obtained by using Taylor's theorem for functions of two variables. Thus we are able to conclude that the solution of $L[u] = 0$ is unique in the vicinity of a curve on which Cauchy data are given, provided det $M \neq 0$.

SAQ 13

Find det M and show that for elliptic equations it is nonzero.

(Solution on p. 33.)

Let us now consider what happens when det $M = 0$, so that M is singular and we cannot solve uniquely for the second derivatives on the curve on which the data are given. Then

$$A\left(\frac{dx}{ds}\right)^2 - B\frac{dx}{ds}\frac{dt}{ds} + C\left(\frac{dt}{ds}\right)^2 = 0.$$

If $dt/ds \neq 0$ throughout the curve then for each value of t there is a unique point on the curve. Thus the curve determines a function

$$t \longmapsto x,$$

and by the chain rule we have

$$\frac{dx}{ds} = \frac{dx}{dt}\frac{dt}{ds}.$$

Thus

$$A\left(\frac{dx}{dt}\right)^2 - B\frac{dx}{dt} + C = 0.$$

A curve satisfying this equation is called a **characteristic (curve)** of the partial differential operator L. (In the special case of the vibrating string operator we obtain the result on *W: page 38*.) If we are given values of u and its first-order partial derivatives on a characteristic, our problem does not possess a unique solution at points off it, however close they may be.

We may solve for dx/dt to obtain

$$\frac{dx}{dt} = \frac{B \pm \sqrt{B^2 - 4AC}}{2A},$$

which is a pair of ordinary differential equations. In a region with $B^2 > 4AC$ (hyperbolic), there are two families of characteristic curves, whilst $B^2 = 4AC$ (parabolic) results in one family, and $B^2 < 4AC$ (elliptic) yields no characteristics at all. In the case where the coefficients are constants we have seen these results in Section 2.2.1. Note that, since A, B and C are now functions the operator L may fall into different classes for different regions.

For hyperbolic operators, i.e. for regions in which an operator has two families of characteristics, we shall employ the theory developed before SAQ 13 to derive a numerical method of solving the Cauchy problem, in Section 2.3. For parabolic and elliptic operators, however, the method fails. This may be explained by virtue of the inapplicability of Cauchy data to these operators since they do not yield well-posed problems (*Unit M201 32*).

READ W: Section 9, pages 44 to 46, ignoring the reference to Section 82.

Notes

(i) *W: page 45, lines 2 to 5*
The two families of characteristics are the curves ξ = constant and η = constant. Differentiating along a curve ξ = constant given, say, by

$$\{(x, t): x = \varphi(s), t = \theta(s), s \in I\}.$$

we get

$$\frac{d\xi}{ds} = \frac{\partial \xi}{\partial x}\frac{dx}{ds} + \frac{\partial \xi}{\partial t}\frac{dt}{ds} = 0$$

and, provided $dt/ds \neq 0$ along the curve,

$$\frac{dx}{dt} = \frac{dx}{ds}\bigg/\frac{dt}{ds} = -\frac{\partial \xi}{\partial t}\bigg/\frac{\partial \xi}{\partial x},$$

from which Equation (9.3) follows. Equation (9.4) is derived similarly.

(ii) *W: page 45, line −12*
Note that this is everywhere hyperbolic.

(iii) *W: page 45, line −6*
This is the solution of

$$\frac{dx}{dt} = 1 + x^2$$

such that $x = a$ when $t = 0$.

(iv) *W: page 46, line 20*
These coefficients have the same sign as A because the polynomial $A\lambda^2 + B\lambda + C$ has no (real) roots by virtue of the ellipticity condition $B^2 - 4AC < 0$.

SAQ 14

W: page 47, Exercise 3

(Solution on p. 33.)

SAQ 15

Classify the equation

$$x^2\frac{\partial^2 u}{\partial t^2} - 2xt\frac{\partial^2 u}{\partial x \partial t} + t^2\frac{\partial^2 u}{\partial x^2} = \frac{x^2}{t}\frac{\partial u}{\partial t} + \frac{t^2}{x}\frac{\partial u}{\partial x} \qquad (x, t) \in R^+ \times R^+,$$

and reduce it to standard form.

Hence obtain a general form for the solution.

(Solution on p. 34.)

2.3 NUMERICAL METHOD OF CHARACTERISTICS

In our discussion of characteristic curves of second-order equations we saw that, if u, $\partial u/\partial x$ and $\partial u/\partial t$ were specified on a curve, then we could evaluate the solution of the hyperbolic differential equation

$$A(x, t)\frac{\partial^2 u}{\partial t^2} + B(x, t)\frac{\partial^2 u}{\partial x \partial t} + C(x, t)\frac{\partial^2 u}{\partial x^2} + F\left(x, t, u, \frac{\partial u}{\partial t}, \frac{\partial u}{\partial x}\right) = 0 \tag{1}$$

at points off the curve (provided that it is not a characteristic) by using Taylor's theorem. The object of this section is to utilize this fact and produce a usable numerical scheme for solving hyperbolic equations with Cauchy data.

The first thing to note is that we cannot, in general, evaluate all the terms of an infinite Taylor series in a finite time! In fact to evaluate more than two is quite tedious; we choose, therefore, the simple approach of using the first-order Taylor approximation (*Unit M201 14, Bilinear and Quadratic Forms*)

$$u\Big|_{(x+\Delta x,\, t+\Delta t)} \simeq u\Big|_{(x,t)} + \Delta x\frac{\partial u}{\partial x}\Big|_{(x,t)} + \Delta t\frac{\partial u}{\partial t}\Big|_{(x,t)}. \tag{2}$$

We must remember that Equation (2) is only an approximation, and to be of any practical use the quantities Δx and Δt must be small. Applying Equation (2) to points along the given initial curve we can evaluate the solution at points off the curve but close to it. If we could also find $p = \partial u/\partial x$ and $q = \partial u/\partial t$ at these points, we would then be in a position to repeat the whole process and generate the solution of the differential equation at points further and further away from the initial curve. A large part of this section deals with finding a way to evaluate p and q at points off the initial curve.

In Section 2.2.2 we derived the system of equations

$$\begin{bmatrix} \frac{dt}{ds} & \frac{dx}{ds} & 0 \\ 0 & \frac{dt}{ds} & \frac{dx}{ds} \\ A & B & C \end{bmatrix} \begin{bmatrix} \frac{\partial^2 u}{\partial t^2} \\ \frac{\partial^2 u}{\partial x \partial t} \\ \frac{\partial^2 u}{\partial x^2} \end{bmatrix} = \begin{bmatrix} \frac{dq}{ds} \\ \frac{dp}{ds} \\ -F \end{bmatrix} \tag{3}$$

for the solution of Equation (1) on the curve $\{(x, t): x = \varphi(s), t = \theta(s), s \in I\}$.

The curve is a characteristic if the determinant of the matrix vanishes at each point, i.e. if

$$A(x, t)\left(\frac{dx}{dt}\right)^2 - B(x, t)\frac{dx}{dt} + C(x, t) = 0. \tag{4}$$

In this case Equation (3) does not have a solution unless the rank of the augmented matrix

$$\begin{bmatrix} \frac{dt}{ds} & \frac{dx}{ds} & 0 & \frac{dq}{ds} \\ 0 & \frac{dt}{ds} & \frac{dx}{ds} & \frac{dp}{ds} \\ A & B & C & -F \end{bmatrix}$$

equals the rank of the matrix M of Equation (3) (see *Unit M100 26, Linear Algebra III*). Now $\det M = 0$ means that

$$\text{rank}\,(M) < 3,$$

so that any three columns are linearly dependent. In particular, this is true of the first, third and fourth columns, and so

$$\det \begin{bmatrix} \dfrac{dt}{ds} & 0 & \dfrac{dq}{ds} \\ 0 & \dfrac{dx}{ds} & \dfrac{dp}{ds} \\ A & C & -F \end{bmatrix} = 0.$$

Hence, we can say that a solution of Equation (3) exists along the characteristics given by Equation (4) only if

$$A\frac{dx}{ds}\frac{dq}{ds} + C\frac{dp}{ds}\frac{dt}{ds} + F\frac{dx}{ds}\frac{dt}{ds} = 0,$$

which, if $dt/ds \neq 0$ along the curve, is equivalent to

$$A\frac{dx}{dt}\frac{dq}{ds} + C\frac{dp}{ds} + F\frac{dx}{ds} = 0. \tag{5}$$

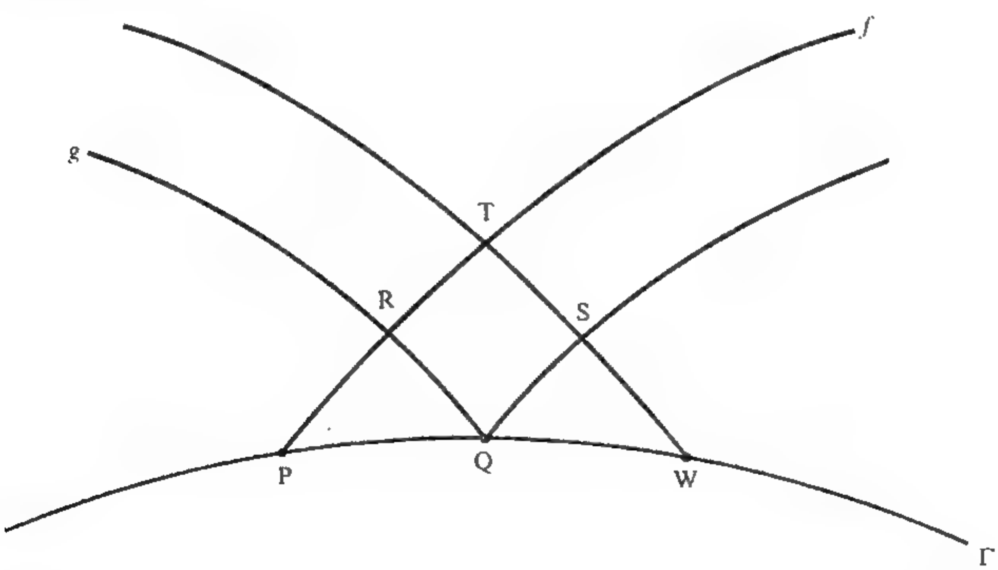
Now for hyperbolic equations, Equation (4) gives two distinct real values for dx/dt. Thus Equation (5) gives rise to two equations from which we might hope to obtain dp/ds and dq/ds, which can then be approximated by finite differences to yield p and q.

Let us write the solutions of Equation (4) as

$$\frac{dx}{dt} = f(x, t) \quad \text{and} \quad \frac{dx}{dt} = g(x, t). \tag{6}$$

It will be convenient to speak of the f- or g-characteristic whenever we wish to discriminate between the two characteristics passing through a given point.

For the remaining part of the argument we shall refer to the diagram, in which Γ is a non-characteristic curve along which initial values for u, p, q are known. P and Q are points on Γ with coordinates (x_P, t_P) and (x_Q, t_Q), which are close together, and the f-characteristic through P intersects the g-characteristic through Q at the point R with coordinates (x_R, t_R).



If Equations (6) cannot be solved analytically—although we should always take advantage of reasonable analytical solutions when available—we can approximate them by using the first-order Taylor approximation once again, giving

$$\left.\begin{aligned} &x_R - x_P = (t_R - t_P)f_P \\ &\text{and} \\ &x_R - x_Q = (t_R - t_Q)g_Q. \end{aligned}\right\} \tag{7}$$

where $f_P = f(x_P, t_P)$ and $g_Q = g(x_Q, t_Q)$. Similar approximations to dp/ds and dq/ds yield the two approximations to Equation (5):

$$A_P f_P \frac{(q_R - q_P)}{(s_R - s_P)} + C_P \frac{(p_R - p_P)}{(s_R - s_P)} + F_P \frac{(x_R - x_P)}{(s_R - s_P)} = 0$$

and

$$A_Q g_Q \frac{(q_R - q_Q)}{(s_R - s_Q)} + C_Q \frac{(p_R - p_Q)}{(s_R - s_Q)} + F_Q \frac{(x_R - x_Q)}{(s_R - s_Q)} = 0$$

where again the subscripts P, Q and R indicate the values of the respective functions and variables at P, Q and R.

Multiplication by $(s_R - s_P)$ and $(s_R - s_Q)$ respectively gives

$$A_P f_P (q_R - q_P) + C_P (p_R - p_P) + F_P (x_R - x_P) = 0 \tag{8a}$$

along the f-characteristic, and

$$A_Q g_Q (q_R - q_Q) + C_Q (p_R - p_Q) + F_Q (x_R - x_Q) = 0 \tag{8b}$$

along the g-characteristic.

Since p and q are assumed known along Γ and hence at P and Q, and since we can calculate x_R from Equations (7), Equations (8) are two simultaneous algebraic equations for p_R and q_R. Hence we have a method for calculating p and q at points off the initial curve and also, using Equation (2), to calculate u there.

This process can be applied to the points Q and W to give the values of u, p and q at the point S. Hence, knowing u, p and q at both R and S, we can apply the procedure once again to evaluate u, p and q at T. Therefore, by taking more points along the initial curve we can progressively estimate the solution of the differential equation at points further away from the initial curve. The points off the initial curve are given by the intersections of the characteristic curves which pass through the chosen points on the initial curve.

Note that in this scheme it does not matter whether we calculate p and q before or after we calculate u at a particular point. However, if we calculate p and q first then we can use an approximation for u which is better than Equation (2). To see this, we first write down the first-order Taylor approximations to $\partial u/\partial x$ and $\partial u/\partial t$ at $(x + \Delta x, t + \Delta t)$:

$$\left.\frac{\partial u}{\partial x}\right|_{(x+\Delta x, t+\Delta t)} \simeq \left.\frac{\partial u}{\partial x}\right|_{(x,t)} + \Delta x \left.\frac{\partial^2 u}{\partial x^2}\right|_{(x,t)} + \Delta t \left.\frac{\partial^2 u}{\partial x \partial t}\right|_{(x,t)} \tag{9a}$$

and

$$\left.\frac{\partial u}{\partial t}\right|_{(x+\Delta x, t+\Delta t)} \simeq \left.\frac{\partial u}{\partial t}\right|_{(x,t)} + \Delta x \left.\frac{\partial^2 u}{\partial x \partial t}\right|_{(x,t)} + \Delta t \left.\frac{\partial^2 u}{\partial t^2}\right|_{(x,t)}. \tag{9b}$$

Multiplying Equation (9a) by $\frac{1}{2}\Delta x$ and Equation (9b) by $\frac{1}{2}\Delta t$, and adding the results, we obtain

$$\frac{1}{2}(\Delta x)^2 \left.\frac{\partial^2 u}{\partial x^2}\right|_{(x,t)} + \Delta x\, \Delta t \left.\frac{\partial^2 u}{\partial x \partial t}\right|_{(x,t)} + \frac{1}{2}(\Delta t)^2 \left.\frac{\partial^2 u}{\partial t^2}\right|_{(x,t)}$$

$$\simeq \frac{1}{2}\Delta x \left[\left.\frac{\partial u}{\partial x}\right|_{(x+\Delta x, t+\Delta t)} - \left.\frac{\partial u}{\partial x}\right|_{(x,t)} \right] + \frac{1}{2}\Delta t \left[\left.\frac{\partial u}{\partial t}\right|_{(x+\Delta x, t+\Delta t)} - \left.\frac{\partial u}{\partial t}\right|_{(x,t)} \right].$$

The left-hand side of this equation is just the expression for the second-order terms in the second-order Taylor approximation:

$$\begin{aligned} u\Big|_{(x+\Delta x,\, t+\Delta t)} &\simeq u\Big|_{(x,t)} + \Delta x\frac{\partial u}{\partial x}\Big|_{(x,t)} + \Delta t\frac{\partial u}{\partial t}\Big|_{(x,t)} + \tfrac{1}{2}(\Delta x)^2\frac{\partial^2 u}{\partial x^2}\Big|_{(x,t)} \\ &\qquad + \Delta x\Delta t\frac{\partial^2 u}{\partial x\partial t}\Big|_{(x,t)} + \tfrac{1}{2}(\Delta t)^2\frac{\partial^2 u}{\partial t^2}\Big|_{(x,t)} \\ &\simeq u\Big|_{(x,t)} + \tfrac{1}{2}\Delta x\left[\frac{\partial u}{\partial x}\Big|_{(x+\Delta x,\, t+\Delta t)} + \frac{\partial u}{\partial x}\Big|_{(x,t)}\right] \\ &\qquad + \tfrac{1}{2}\Delta t\left[\frac{\partial u}{\partial t}\Big|_{(x+\Delta x,\, t+\Delta t)} + \frac{\partial u}{\partial t}\Big|_{(x,t)}\right]. \end{aligned}$$

This apparently first-order approximation is more accurate than Equation (2) since it incorporates the second-order terms of the Taylor expansion. Applying this approximation to the points P, Q and R gives the approximations

$$u_R \simeq u_P + \tfrac{1}{2}(p_R + p_P)(x_R - x_P) + \tfrac{1}{2}(q_R + q_P)(t_R - t_P)$$

and

$$u_R \simeq u_Q + \tfrac{1}{2}(p_R + p_Q)(x_R - x_Q) + \tfrac{1}{2}(q_R + q_Q)(t_R - t_Q).$$

We can use either of the last two equations to calculate u_R, the remaining equation being used as a check on the method.

We summarize the method as follows. Given a second-order hyperbolic linear partial differential equation in u with (initial) values of u, $\partial u/\partial x$, $\partial u/\partial t$ given on a non-characteristic curve, the solution u can be found at points off the initial curve by repeating the following calculations:

Stage (i)

calculate the coordinates of the point of intersection of two characteristic curves given by (x_R, t_R) from

$$x_R - x_P = (t_R - t_P)f_P,$$

$$x_R - x_Q = (t_R - t_Q)g_Q,$$

where $f(x, t)$ and $g(x, t)$ are solutions of the equation

$$A(x,t)\left(\frac{dx}{dt}\right)^2 - B(x,t)\frac{dx}{dt} + C(x,t) = 0;$$

Stage (ii)

calculate p_R and q_R from

$$A_P f_P(q_R - q_P) + C_P(p_R - p_P) + F_P(x_R - x_P) = 0$$

and

$$A_Q g_Q(q_R - q_Q) + C_Q(p_R - p_Q) + F_Q(x_R - x_Q) = 0;$$

Stage (iii)

calculate u_R from both

$$u_R = u_P + \tfrac{1}{2}(p_R + p_P)(x_R - x_P) + \tfrac{1}{2}(q_R + q_P)(t_R - t_P)$$

and

$$u_R = u_Q + \tfrac{1}{2}(p_R + p_Q)(x_R - x_Q) + \tfrac{1}{2}(q_R + q_Q)(t_R - t_Q),$$

the second calculation being used as a check on the method.

Example (*S: page 117, Exercise 1*)

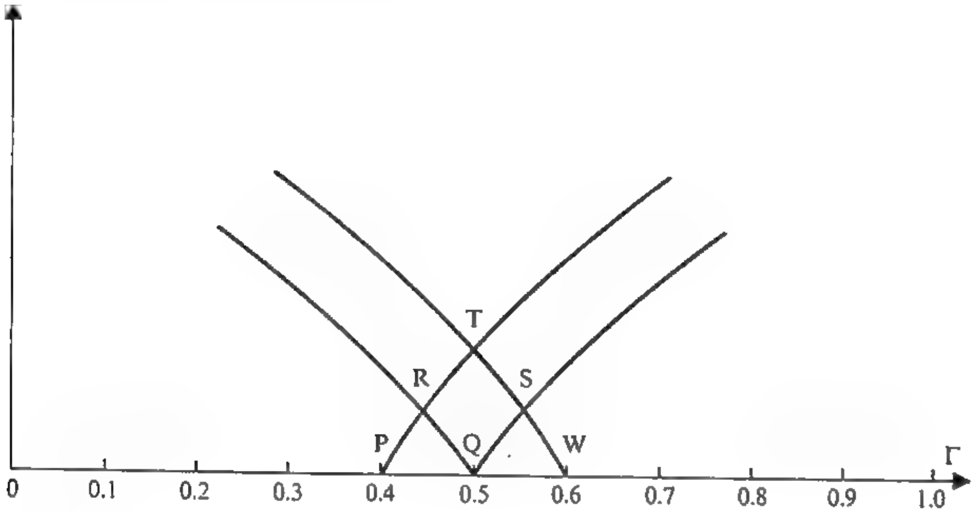
The equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial x\partial t} - 2\frac{\partial^2 u}{\partial t^2} + 1 = 0 \qquad 0 < x < 1,\ t > 0$$

satisfies the initial conditions

$$u = \frac{\partial u}{\partial t} = x \text{ along } t = 0, 0 \leqslant x \leqslant 1.$$

Use the numerical method of characteristics to calculate the solution at the points R, S and T shown on the diagram. The coordinates of the initial points P, Q and W are (0.4,0), (0.5,0) and (0.6,0) respectively.



Solution

The differential equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial x \partial t} - 2\frac{\partial^2 u}{\partial t^2} + 1 = 0$$

has characteristics specified by the roots of the equation

$$-2\left(\frac{dx}{dt}\right)^2 - \frac{dx}{dt} + 1 = 0,$$

which are

$$\frac{dx}{dt} = \tfrac{1}{2}, -1.$$

The characteristics are given by

$$2x - t = \text{constant},$$

$$t + x = \text{constant}.$$

Stage (i)

We calculate x_R and t_R from

$$2x_R - t_R = 2x_P - t_P,$$

$$t_R + x_R = t_Q + x_Q.$$

Stage (ii)

p_R and q_R are obtained from the equations

$$-(q_R - q_P) + (p_R - p_P) + (x_R - x_P) = 0,$$

$$2(q_R - q_Q) + (p_R - p_Q) + (x_R - x_Q) = 0.$$

The values of p, q at P and Q are obtained from

$$q(x, 0) = \frac{\partial u}{\partial t}(x, 0) = x \qquad 0 \leqslant x \leqslant 1,$$

and

$$p(x, 0) = \frac{\partial u}{\partial x}(x, 0) = \frac{du}{dx}(x, 0) \text{ along } t = 0$$

$$= 1 \qquad 0 \leqslant x \leqslant 1.$$

Stage (iii)

The formula(s) for u_R can now be used since we have already determined x, t, p, q at the points P, Q and R. The values of u at P and Q are given by the condition

$$u(x, 0) = x \qquad 0 \leqslant x \leqslant 1.$$

We now repeat the whole process for the points S and T, obtaining the data set out in the following table.

	x	t	p	q	u	exact value of u
P	0.4	0.0	1.0	0.4	0.4	0.4
Q	0.5	0.0	1.0	0.5	0.5	0.5
W	0.6	0.0	1.0	0.6	0.6	0.6
R	$\frac{13}{30}$	$\frac{2}{30}$	$\frac{32}{30}$	$\frac{15}{30}$	$\frac{418}{900}$	$\frac{418}{900}$
S	$\frac{16}{30}$	$\frac{2}{30}$	$\frac{32}{30}$	$\frac{18}{30}$	$\frac{514}{900}$	$\frac{514}{900}$
T	$\frac{14}{30}$	$\frac{4}{30}$	$\frac{34}{30}$	$\frac{18}{30}$	$\frac{484}{900}$	$\frac{484}{900}$

The exact values of u have been determined from the analytical solution

$$u(x, t) = x + xt + t^2/2$$

from which we obtain

$$p(x, t) = 1 + t \text{ and } q(x, t) = x + t.$$

The precision of our numerical solution is due to the fact that the quadratic approximation is exact.

SAQ 16

Use the numerical method of characteristics to obtain $u\,(0.15, 0.15)$, where

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2} \qquad 0 < x < 1, \quad t > 0,$$

subject to the boundary conditions

$$u(0, t) = u(1, t) = 0 \qquad t \geqslant 0,$$

and the initial conditions

$$u(x, 0) = \tfrac{1}{2}x(1 - x) \qquad 0 \leqslant x \leqslant 1,$$

$$\frac{\partial u}{\partial t}(x, 0) = 0 \qquad 0 \leqslant x \leqslant 1.$$

(Solution on p. 35.)

2.4 SUMMARY

In this unit, we have looked at linear partial differential equations of the second order, subject to linear subsidiary conditions. We showed how their linear properties could be used to give us the *principle of superposition*, and also to examine the *uniqueness* of solutions to these problems.

Second-order operators of the general form

$$A\frac{\partial^2 u}{\partial t^2} + B\frac{\partial^2 u}{\partial x \partial t} + C\frac{\partial^2 u}{\partial x^2} + F\left(x, t, u, \frac{\partial u}{\partial t}, \frac{\partial u}{\partial x}\right) = 0$$

are classified as *hyperbolic* ($B^2 > 4AC$), *parabolic* ($B^2 = 4AC$) and *elliptic* ($B^2 < 4AC$). We discovered that for hyperbolic equations, if x, t, u, and $\partial u/\partial x$ or $\partial u/\partial t$ are specified on a given curve Γ, our problem possesses a unique solution throughout the region bounded by Γ and a pair of characteristics.

We found that, along the *characteristics*, x and t are related by

$$A\left(\frac{dx}{dt}\right)^2 - B\frac{dx}{dt} + C = 0,$$

and saw how the equations of the characteristics can be used to find a coordinate transformation reducing the differential operator to *standard form*.

Finally, we saw the use of characteristics in obtaining numerical solutions of hyperbolic equations.

2.5 SOLUTIONS TO SELF-ASSESSMENT QUESTIONS

Solution to SAQ 1

We are given

$$u = p(x^2 - t) + q(x^2 + t).$$

So,

$$\frac{\partial u}{\partial x} = 2x[p'(x^2 - t) + q'(x^2 + t)],$$

$$\frac{\partial^2 u}{\partial x^2} = 4x^2[p''(x^2 - t) + q''(x^2 + t)] + 2[p'(x^2 - t) + q'(x^2 + t)],$$

and

$$\frac{\partial^2 u}{\partial t^2} = p''(x^2 - t) + q''(x^2 + t).$$

Hence,

$$\frac{\partial^2 u}{\partial x^2} - \frac{1}{x}\frac{\partial u}{\partial x} - 4x^2\frac{\partial^2 u}{\partial t^2} = 0.$$

Alternatively, if you are feeling energetic, you can multiply out the determinant of the matrix on p. 7 and set it equal to zero.

Solution to SAQ 2

If

$$u = p(x + \alpha t + i\beta y) + q(x - \alpha t - i\beta y),$$

then

$$\frac{\partial^2 u}{\partial x^2} = p''(x + \alpha t + i\beta y) + q''(x - \alpha t - i\beta y),$$

$$\frac{\partial^2 u}{\partial y^2} = -\beta^2[p''(x + \alpha t + i\beta y) + q''(x - \alpha t - i\beta y)],$$

$$\frac{\partial^2 u}{\partial t^2} = \alpha^2[p''(x + \alpha t + i\beta y) + q''(x - \alpha t - i\beta y)].$$

Hence,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = (1 - \beta^2)[p''(x + \alpha t + i\beta y) + q''(x - \alpha t - i\beta y)]$$

$$= \frac{\partial^2 u}{\partial t^2},$$

provided $1 - \beta^2 = \alpha^2$ or $\alpha^2 + \beta^2 = 1$.

Solution to SAQ 3

We have

$$u^2 = p(\xi) + q(\eta).$$

We may differentiate once, using the chain rule, to obtain

$$2u\frac{\partial u}{\partial x} = p'(\xi)\frac{\partial \xi}{\partial x} + q'(\eta)\frac{\partial \eta}{\partial x}$$

and

$$2u\frac{\partial u}{\partial t} = p'(\xi)\frac{\partial \xi}{\partial t} + q'(\eta)\frac{\partial \eta}{\partial t}.$$

Further differentiation of these equations yields

$$2\left(\frac{\partial u}{\partial x}\right)^2 + 2u\frac{\partial^2 u}{\partial x^2} = p''(\xi)\left(\frac{\partial \xi}{\partial x}\right)^2 + p'(\xi)\frac{\partial^2 \xi}{\partial x^2} + q''(\eta)\left(\frac{\partial \eta}{\partial x}\right)^2 + q'(\eta)\frac{\partial^2 \eta}{\partial x^2},$$

$$2\frac{\partial u}{\partial x}\frac{\partial u}{\partial t} + 2u\frac{\partial^2 u}{\partial x \partial t} = p''(\xi)\frac{\partial \xi}{\partial x}\frac{\partial \xi}{\partial t} + p'(\xi)\frac{\partial^2 \xi}{\partial x \partial t} + q''(\eta)\frac{\partial \eta}{\partial x}\frac{\partial \eta}{\partial t} + q'(\eta)\frac{\partial^2 \eta}{\partial x \partial t},$$

and

$$2\left(\frac{\partial u}{\partial t}\right)^2 + 2u\frac{\partial^2 u}{\partial t^2} = p''(\xi)\left(\frac{\partial \xi}{\partial t}\right)^2 + p'(\xi)\frac{\partial^2 \xi}{\partial t^2} + q''(\eta)\left(\frac{\partial \eta}{\partial t}\right)^2 + q'(\eta)\frac{\partial^2 \eta}{\partial t^2},$$

for all values of x and t such that the second partial derivatives exist. These five equations may be expressed in matrix form as

$$\begin{bmatrix} 0 & \frac{\partial \xi}{\partial x} & 0 & \frac{\partial \eta}{\partial x} & -2u\frac{\partial u}{\partial x} \\ 0 & \frac{\partial \xi}{\partial t} & 0 & \frac{\partial \eta}{\partial t} & -2u\frac{\partial u}{\partial t} \\ \left(\frac{\partial \xi}{\partial x}\right)^2 & \frac{\partial^2 \xi}{\partial x^2} & \left(\frac{\partial \eta}{\partial x}\right)^2 & \frac{\partial^2 \eta}{\partial x^2} & -2\left(\frac{\partial u}{\partial x}\right)^2 - 2u\frac{\partial^2 u}{\partial x^2} \\ \frac{\partial \xi}{\partial x}\frac{\partial \xi}{\partial t} & \frac{\partial^2 \xi}{\partial x \partial t} & \frac{\partial \eta}{\partial x}\frac{\partial \eta}{\partial t} & \frac{\partial^2 \eta}{\partial x \partial t} & -2\frac{\partial u}{\partial x}\frac{\partial u}{\partial t} - 2u\frac{\partial^2 u}{\partial x \partial t} \\ \left(\frac{\partial \xi}{\partial t}\right)^2 & \frac{\partial^2 \xi}{\partial t^2} & \left(\frac{\partial \eta}{\partial t}\right)^2 & \frac{\partial^2 \eta}{\partial t^2} & -2\left(\frac{\partial u}{\partial t}\right)^2 - 2u\frac{\partial^2 u}{\partial t^2} \end{bmatrix} \begin{bmatrix} p''(\xi) \\ p'(\xi) \\ q''(\eta) \\ q'(\eta) \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

As in the text (Section 2.1.1), the existence of the nonzero column vector $(p''(\xi), p'(\xi), q''(\eta), q'(\eta), 1)$ satisfying this equation means that the determinant of the matrix must be equal to zero.

On expansion, this results in an equation of the form

$$A(x, t)u\frac{\partial u}{\partial x} + B(x, t)u\frac{\partial u}{\partial t} + C(x, t)\left[\left(\frac{\partial u}{\partial x}\right)^2 + u\frac{\partial^2 u}{\partial x^2}\right]$$
$$+ D(x, t)\left[\frac{\partial u}{\partial x}\frac{\partial u}{\partial t} + u\frac{\partial^2 u}{\partial x \partial t}\right] + E(x, t)\left[\left(\frac{\partial u}{\partial t}\right)^2 + u\frac{\partial^2 u}{\partial t^2}\right] = 0,$$

where A, B, C, D and E are known functions of two variables. This equation is non-linear unless C, D and E are the zero functions, in which case it is first-order.

Solution to SAQ 4

A particular solution of the differential equation is the steady-state solution, which satisfies

$$v(x, t) = v(x, 0) \qquad 0 \leqslant x \leqslant \pi, \ t \geqslant 0$$

and

$$-\frac{\partial^2 v}{\partial x^2}(x, t) = 2\cos x - x\sin x \qquad 0 < x < \pi, \ t > 0.$$

Integration yields

$$-\frac{\partial v}{\partial x}(x, t) = 2\sin x + x\cos x - \sin x$$
$$= \sin x + x\cos x \qquad 0 < x < \pi, \ t > 0$$

and

$$-v(x, t) = -\cos x + x\sin x + \cos x$$
$$= x\sin x \qquad 0 < x < \pi, \ t > 0.$$

We have chosen constants of integration such that v satisfies the boundary conditions on u.

Next we let $w = u - v$: then w must satisfy

$$\frac{\partial^2 w}{\partial t^2} - \frac{\partial^2 w}{\partial x^2} = 0 \qquad 0 < x < \pi,\ t > 0,$$

$$w(x, 0) = x \sin x \qquad 0 \leqslant x \leqslant \pi,$$

$$\frac{\partial w}{\partial t}(x, 0) = 0 \qquad 0 \leqslant x \leqslant \pi,$$

$$w(0, t) = 0 \qquad t \geqslant 0,$$

$$w(\pi, t) = 0 \qquad t \geqslant 0.$$

By Equation (2.7) on *W: page 10*, we have

$$w(x, t) = \tfrac{1}{2}[(x + t)\sin(x + t) + (x - t)\sin(x - t)]$$

in the triangular region

$$t \leqslant x, \quad t \leqslant \pi - x, \quad t \geqslant 0.$$

Now

$$u(x, t) = w(x, t) - x \sin x:$$

hence

$$u\left(\frac{\pi}{2}, \frac{\pi}{2}\right) = -\frac{\pi}{2}.$$

In fact, this result is independent of the boundary conditions at $x = 0$ and $x = \pi$.

Solution to SAQ 5

C and **D**.

A is nonlinear owing to the presence of the term $u\dfrac{\partial u}{\partial x}$, **B** owing to $\left(\dfrac{\partial u}{\partial t}\right)^2$ and **E** owing to $\cos u$.

Solution to SAQ 6

The point $(\frac{1}{2}, \frac{1}{6}\pi)$ is outside the triangle in which the solution is determined by the initial conditions. However, we cannot apply the formula solution—Equation (2.16) on *W: page 13*—directly because the differential equation is nonhomogeneous and the boundary condition at $x = 0$ is neither fixed nor free. We proceed as in the example on *W: page 32* and define the function v, which satisfies the last two conditions, by

$$v : (x, t) \longmapsto (1 - x)\sin^2 t \qquad 0 \leqslant x \leqslant 1, t \geqslant 0.$$

Putting $w = u - v$, we obtain

$$\frac{\partial^2 w}{\partial t^2} - \frac{\partial^2 w}{\partial x^2} = \frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} - 2(1 - x)\cos 2t$$

$$= 0 \qquad 0 < x < 1,\ t > 0,$$

$$w(x, 0) = 0 \qquad 0 \leqslant x \leqslant 1,$$

$$\frac{\partial w}{\partial t}(x, 0) = 0 \qquad 0 \leqslant x \leqslant 1,$$

$$w(0, t) = 0 \qquad t \geqslant 0,$$

$$w(1, t) = 0 \qquad t \geqslant 0,$$

whose solution is clearly

$$w(x, t) = 0 \qquad 0 \leqslant x \leqslant 1,\ t \geqslant 0.$$

Hence

$$u(x, t) = (1 - x)\sin^2 t \qquad 0 \leqslant x \leqslant 1, \ t \geqslant 0.$$

We have

$$\sin \tfrac{1}{6}\pi = \tfrac{1}{2},$$

and so

$$u(\tfrac{1}{2}, \tfrac{1}{6}\pi) = \tfrac{1}{8}.$$

Solution to SAQ 7

Let u_1 and u_2 be two solutions and put $v = u_1 - u_2$. Then

$$\frac{\partial^2 v}{\partial t^2} - c^2\frac{\partial^2 v}{\partial x^2} = 0 \qquad 0 < x < l, \ t > 0.$$

$$v(x, 0) = \frac{\partial v}{\partial t}(x, 0) = 0 \qquad 0 \leqslant x \leqslant l,$$

$$v(0, t) = v(l, t) = 0 \qquad t \geqslant 0.$$

The formula solution—Equation (2.16) on *W: page 13*—gives $v(x, t) = 0$ so that $u_1 = u_2$. Thus there is at most one solution to the original problem.

Solution to SAQ 8

Equations (7.7) on *W: page 38* give the characteristics in the form

$$C_1 : t = \bar{t} - \int_x^{\bar{x}} \frac{d\xi}{1 + \xi^2}$$

and

$$C_2 : t = \bar{t} - \int_{\bar{x}}^{x} \frac{d\xi}{1 + \xi^2},$$

where $(\bar{x}, \bar{t}) = (\tfrac{1}{4}, 3)$.

Thus, on C_1 we have

$$t = 3 - \tan^{-1}\tfrac{1}{4} + \tan^{-1}x$$

and on C_2 we have

$$t = 3 + \tan^{-1}\tfrac{1}{4} - \tan^{-1}x.$$

Therefore the domain of dependence of $(\tfrac{1}{4}, 3)$ is the region determined by

$$0 \leqslant x \leqslant \tfrac{1}{4}, \quad 0 \leqslant t \leqslant 3 - \tan^{-1}\tfrac{1}{4} + \tan^{-1}x$$

and

$$\tfrac{1}{4} \leqslant x \leqslant 1, \quad 0 \leqslant t \leqslant 3 + \tan^{-1}\tfrac{1}{4} - \tan^{-1}x.$$

Solution to SAQ 9

Applying the same method as that used in the solution to SAQ 8 we get

$$C_1 : t = \tan^{-1}x - \tan^{-1}\tfrac{1}{2}$$

and

$$C_2 : t = \tan^{-1}\tfrac{1}{2} - \tan^{-1}x.$$

The domain of influence of $(\tfrac{1}{2}, 0)$ is determined by the two upwards characteristics and is given by

$$0 \leqslant x \leqslant \tfrac{1}{2}, \quad t \geqslant \tan^{-1}\tfrac{1}{2} - \tan^{-1}x$$

and

$$\tfrac{1}{2} \leqslant x \leqslant 1, \quad t \geqslant \tan^{-1}x - \tan^{-1}\tfrac{1}{2}.$$

Solution to SAQ 10

We have

$$x' = x - \frac{B}{2A}t \text{ and } t' = t.$$

Therefore

$$\frac{\partial x'}{\partial x} = 1, \frac{\partial x'}{\partial t} = -\frac{B}{2A},$$

$$\frac{\partial t'}{\partial x} = 0, \frac{\partial t'}{\partial t} = 1.$$

Differentiating u according to the chain rule for several variables, we obtain

$$\frac{\partial u}{\partial t} = -\frac{B}{2A}\frac{\partial u}{\partial x'} + \frac{\partial u}{\partial t'},$$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial x'},$$

$$\frac{\partial^2 u}{\partial t^2} = \frac{B^2}{4A^2}\frac{\partial^2 u}{\partial x'^2} - \frac{B}{A}\frac{\partial^2 u}{\partial x'\partial t'} + \frac{\partial^2 u}{\partial t'^2},$$

$$\frac{\partial^2 u}{\partial x\partial t} = -\frac{B}{2A}\frac{\partial^2 u}{\partial x'^2} + \frac{\partial^2 u}{\partial x'\partial t'},$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial x'^2}.$$

In terms of the new coordinates we obtain

$$\begin{aligned} L[u] &= \frac{B^2}{4A}\frac{\partial^2 u}{\partial x'^2} - B\frac{\partial^2 u}{\partial x'\partial t'} + A\frac{\partial^2 u}{\partial t'^2} - \frac{B^2}{2A}\frac{\partial^2 u}{\partial x'^2} + B\frac{\partial^2 u}{\partial x'\partial t'} + C\frac{\partial^2 u}{\partial x'^2} \\ &= A\frac{\partial^2 u}{\partial t'^2} - \left(\frac{B^2}{4A} - C\right)\frac{\partial^2 u}{\partial x'^2}. \end{aligned}$$

Since $B^2 - 4AC > 0$, this has the form of the wave operator,

$$L[u] = A\left[\frac{\partial^2 u}{\partial t'^2} - c^2\frac{\partial^2 u}{\partial x'^2}\right]$$

with

$$c^2 = \frac{B^2 - 4AC}{4A^2}.$$

Solution to SAQ 11

(a) $B^2 - 4AC = -3$. The operator is elliptic and has no characteristics.

(b) $B^2 - 4AC = 0$. The operator is parabolic and the characteristic through $(0, 1)$ is

$$x - 2t + 2 = 0.$$

(c) $B^2 - 4AC = 12$. The operator is hyperbolic and the characteristics through $(0, 1)$ are

$$x + (2 + \sqrt{3})(t - 1) = 0$$

and

$$x + (2 - \sqrt{3})(t - 1) = 0.$$

Solution to SAQ 12

We have seen in the solution to SAQ 11 that the operator is parabolic. By the result on *W: page 42* the transformation

$$\xi = 2x - 4t$$

$$\eta = t$$

transforms the equation into the standard form

$$\frac{\partial^2 u}{\partial \eta^2} = 0,$$

whose solution is given by

$$u = p(\xi) + \eta q(\xi).$$

Thus

$$u = p(2x - 4t) + tq(2x - 4t),$$

where p and q are arbitrary functions of one variable.

Solution to SAQ 13

We obtain

$$\det M = A\left(\frac{dx}{ds}\right)^2 - B\frac{dx}{ds}\frac{dt}{ds} + C\left(\frac{dt}{ds}\right)^2.$$

For an elliptic equation $4AC - B^2 > 0$; hence $A \neq 0$. Therefore

$$\det M = A\left(\frac{dx}{ds}\right)^2 - B\frac{dx}{ds}\frac{dt}{ds} + \frac{B^2}{4A}\left(\frac{dt}{ds}\right)^2 - \frac{B^2}{4A}\left(\frac{dt}{ds}\right)^2 + C\left(\frac{dt}{ds}\right)^2$$

$$= A\left[\frac{dx}{ds} - \frac{B}{2A}\frac{dt}{ds}\right]^2 + \left(C - \frac{B^2}{4A}\right)\left(\frac{dt}{ds}\right)^2$$

$$\neq 0,$$

since $4AC - B^2 > 0$ and dx/ds, dt/ds are not simultaneously zero at any point.

Solution to SAQ 14

Using the method of *W: page 45*, the characteristics are determined by

$$\frac{dx}{dt} = \tfrac{1}{2}(e^x + t^2) \pm \tfrac{1}{2}[(e^x + t^2)^2 - 4t^2e^x]$$

$$= \tfrac{1}{2}[e^x + t^2 \pm (e^x - t^2)]$$

$$= e^x \text{ or } t^2.$$

The condition

$$dx/dt = e^x$$

gives

$$\int dt = \int e^{-x}\, dx,$$

and

$$t = c_1 - e^{-x}.$$

Let $x = a$ when $t = 0$: then

$$c_1 = e^{-a},$$

and hence

$$a = -\ln(e^{-x} + t).$$

We therefore take

$$\xi = -\ln(e^{-x} + t).$$

The condition $dx/dt = t^2$ gives

$$x = \tfrac{1}{3}t^3 + c_2.$$

Let $x = a$ when $t = 0$: then

$$c_2 = a,$$

and hence

$$a = x - \tfrac{1}{3}t^3.$$

We therefore take

$$\eta = x - \tfrac{1}{3}t^3.$$

Solution to SAQ 15

$$(2xt)^2 - 4x^2t^2 = 0,$$

and so the equation is parabolic throughout its domain. Along a characteristic

$$\frac{dx}{dt} = -\frac{t}{x},$$

and

$$x^2 = -t^2 + \text{constant}.$$

So let

$$\xi = x^2 + t^2,$$

and take

$$\eta = t.$$

We obtain

$$\frac{\partial u}{\partial t} = 2t\frac{\partial u}{\partial \xi} + \frac{\partial u}{\partial \eta},$$

$$\frac{\partial u}{\partial x} = 2x\frac{\partial u}{\partial \xi},$$

$$\frac{\partial^2 u}{\partial t^2} = 2\frac{\partial u}{\partial \xi} + 4t^2\frac{\partial^2 u}{\partial \xi^2} + 4t\frac{\partial^2 u}{\partial \xi \partial \eta} + \frac{\partial^2 u}{\partial \eta^2},$$

$$\frac{\partial^2 u}{\partial x \partial t} = 4xt\frac{\partial^2 u}{\partial \xi^2} + 2x\frac{\partial^2 u}{\partial \xi \partial \eta},$$

$$\frac{\partial^2 u}{\partial x^2} = 2\frac{\partial u}{\partial \xi} + 4x^2\frac{\partial^2 u}{\partial \xi^2}.$$

The equation now becomes

$$x^2\left[2\frac{\partial u}{\partial \xi} + 4t^2\frac{\partial^2 u}{\partial \xi^2} + 4t\frac{\partial^2 u}{\partial \xi \partial \eta} + \frac{\partial^2 u}{\partial \eta^2}\right] - 2xt\left[4xt\frac{\partial^2 u}{\partial \xi^2} + 2x\frac{\partial^2 u}{\partial \xi \partial \eta}\right]$$
$$+ t^2\left[2\frac{\partial u}{\partial \xi} + 4x^2\frac{\partial^2 u}{\partial \xi^2}\right] = \frac{x^2}{t}\left[2t\frac{\partial u}{\partial \xi} + \frac{\partial u}{\partial \eta}\right] + \frac{t^2}{x}2x\frac{\partial u}{\partial \xi}$$

which simplifies to

$$x^2\frac{\partial^2 u}{\partial \eta^2} = \frac{x^2}{t}\frac{\partial u}{\partial \eta}.$$

Since $x \neq 0$ we may write this as

$$\frac{\partial^2 u}{\partial \eta^2} - \frac{1}{\eta}\frac{\partial u}{\partial \eta} = 0$$

or

$$\frac{\partial}{\partial \eta}\left(\frac{1}{\eta}\frac{\partial u}{\partial \eta}\right) = 0;$$

$$\therefore \frac{1}{\eta}\frac{\partial u}{\partial \eta} = f_1(\xi)$$

or

$$\frac{\partial u}{\partial \eta} = \eta f_1(\xi);$$

$$\therefore u = \tfrac{1}{2}\eta^2 f_1(\xi) + f_2(\xi).$$

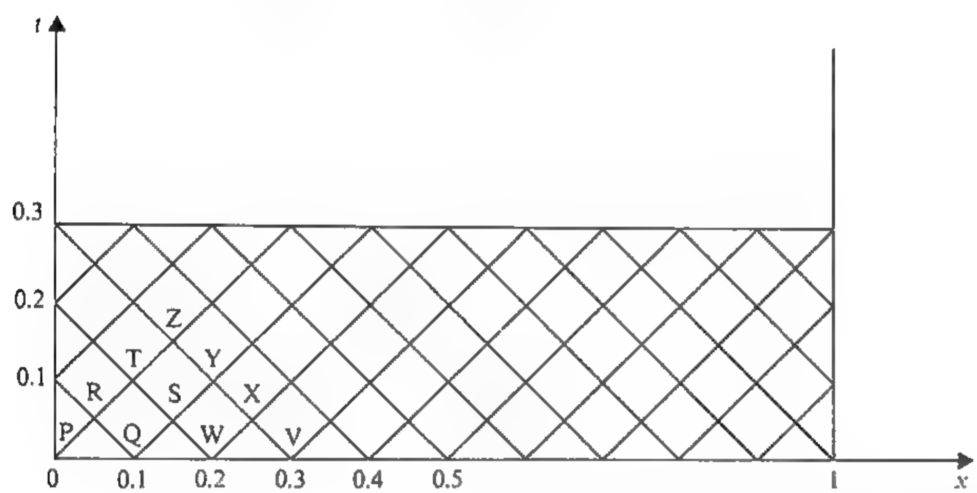
Thus the general solution is

$$u = \tfrac{1}{2}t^2 f_1(x^2 + t^2) + f_2(x^2 + t^2).$$

Solution to SAQ 16

The characteristics for the operator are

$$t + x = \text{constant and } t - x = \text{constant.}$$



Let P, Q, W, V, R, S, X, T, Y, Z be as shown in the diagram. The lines shown are the characteristics. We calculate u_R as follows.

Stage (i)

We calculate x_R and t_R from

$$t_R - x_R = t_P - x_P,$$

$$t_R + x_R = t_Q + x_Q.$$

Stage (ii)

p_R and q_R are obtained from

$$q_R - q_P - (p_R - p_P) = 0,$$

$$-(q_R - q_Q) - (p_R - p_Q) = 0.$$

The values of p, q at P and Q are obtained from

$$q(x, 0) = \frac{\partial u}{\partial t}(x, 0) = 0 \qquad 0 \leqslant x \leqslant 1$$

and

$$p(x, 0) = \frac{\partial u}{\partial x}(x, 0) = \frac{du}{dx}(x, 0) \qquad \text{along } t = 0$$

$$= -x + \tfrac{1}{2}.$$

Stage (iii)

u_R may now be determined using the values obtained for x_R, t_R, p_R, q_R. u_P and u_Q are given by the condition

$$u(x, 0) = \tfrac{1}{2}x(1 - x) \qquad 0 \leqslant x \leqslant 1.$$

We repeat the whole process for the other points, obtaining the following table.

	x	t	p	q	u	analytical solution	
P	0	0	0.5	0	0	0	Initial values
Q	0.1	0	0.4	0	0.045	0.045	
W	0.2	0	0.3	0	0.080	0.080	
V	0.3	0	0.2	0	0.105	0.105	
R	0.05	0.05	0.45	−0.05	0.0225	0.022	
S	0.15	0.05	0.35	−0.05	0.0625	0.062	
X	0.25	0.05	0.25	−0.05	0.0925	0.092	
T	0.1	0.1	0.4	−0.1	0.040	0.040	
Y	0.2	0.1	0.3	−0.1	0.075	0.075	
Z	0.15	0.15	0.35	−0.15	0.0525	0.052	

The analytical solution is

$$u(x, t) = \frac{2}{\pi^3} \sum_{n=1}^{\infty} \frac{1}{n^3}(1 - \cos n\pi) \cos n\pi t \sin n\pi x.$$

The first three terms of this series have been used for the comparison column in the table.

In this question we have not had to use the given boundary conditions since we have calculated the solution at points where the solution depends solely on the initial data. However, if, in this question, we had wanted the solution at the point (0.05, 0.15) we would have needed to use the given boundary condition along $x = 0$. The chosen characteristics intersect each other at points on the boundaries $x = 0$ and $x = 1$ and therefore it is possible to "pick-up" these known values and use them in the numerical scheme. (If the chosen characteristics do not intersect at points on the boundaries we could still use the method of characteristics in the regions affected by the boundary values by choosing a new set of characteristics there.)

Unit 3 Elliptic and Parabolic Equations

Contents

Set Books

G. D. Smith, *Numerical Solution of Partial Differential Equations* (Oxford, 1971).

H. F. Weinberger, *A First Course in Partial Differential Equations* (Blaisdell, 1965).

It is essential to have these books; the course is based on them and will not make sense without them. They are referred to in the text as ***S*** and ***W*** respectively.

Unit 3 is based on ***W***: Chapter III, Sections 10 to 13.

Conventions

Before working through this text make sure you have read *A Guide to the Course: Partial Differential Equations of Applied Mathematics.* References to Open University courses in mathematics take the form:

Unit M100 13, Integration II for the Mathematics Foundation Course,
Unit M201 23, The Wave Equation for the Linear Mathematics Course.

Bibliography

M. R. Spiegel, *Vector Analysis* (Schaum, 1966). The first two chapters contain some basic material about vectors. Chapter 4 discusses scalar and vector fields and introduces the gradient and divergence operators. Surface and volume integrals are covered in Chapter 5 and the Divergence Theorem is treated in Chapter 6.

C. A. Coulson, *Waves* (Oliver and Boyd, 1955). You may find this book generally useful as it gives many examples of physical situations which give rise to the wave equation. The derivation of the membrane equation appears in Chapter III.

3.0 INTRODUCTION

This unit is concerned with the general properties of two important partial differential equations which occur frequently in applications to physics and engineering.

The first is *Poisson's equation* in two or three dimensions,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = -F,$$

or

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = -F.$$

Poisson's equation is the simplest example of an *elliptic* equation, and an important application of it is discussed in Section 3.2.2 for the flow of a viscous fluid in a pipe. The special case in which F is the zero function is *Laplace's equation*, which has many applications including:

(a) gravitational potential;

(b) steady heat flow;

(c) flows of non-viscous fluids;

(d) electrostatics;

(e) surface waves on liquids; and

(f) elasticity.

Uniqueness and maximum values of the solutions are discussed together with an explanation of what we mean by a *properly posed* problem. Solutions to Poisson's equation can be obtained analytically for simple boundaries and boundary conditions only. Section 3.2 introduces extremum principles, which can be used for obtaining numerical upper and lower bounds for certain integrals involving the solution. Bounds for the flow of a viscous liquid in a pipe are presented in detail.

The second partial differential equation which we discuss is the *diffusion equation*, which for three spatial dimensions has the form

$$\frac{\partial u}{\partial t} - k\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) = 0.$$

It is an example of a *parabolic equation*, and one of its applications, which we model in this unit, is to the time-dependent conduction of heat. We discuss both the uniqueness of the solution and a maximum principle for this equation.

We include some results in vector calculus where these are necessary for our development of the material; and some elementary facts about vectors are collected in the Appendix.

3.1 POISSON'S EQUATION

3.1.1 The Membrane Equation

In order to discuss problems in two and three spatial dimensions, it is convenient to employ vector notation; that is, we describe physical quantities (where appropriate) by geometric vectors, or by corresponding ordered pairs or triples. To remind you of the sort of vector algebra which we use we have set SAQ 1; refer to the Appendix if you are unsuccessful in interpreting it.

SAQ 1

If $\mathbf{a} = \mathbf{i} + \mathbf{j} + \mathbf{k}$, $\mathbf{b} = 2\mathbf{i} - \mathbf{j} - \mathbf{k}$ and $\mathbf{c} = -\mathbf{i} + 3\mathbf{j} + \mathbf{k}$, find

(i) $|\mathbf{a}|$

(ii) $|\mathbf{a} + 2\mathbf{b}|$

(iii) $\mathbf{a} \cdot \mathbf{b}$

(iv) $\mathbf{a} \times \mathbf{b}$

(v) $\mathbf{a} \cdot \mathbf{b} \times \mathbf{c}$

(vi) $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$.

(Solution on p. 34.)

We now proceed to the definitions of vector calculus which are required to enable the use of vector quantities in differential equations. In applications our domain of interest may be the plane R^2 or three-dimensional space R^3.

Let D denote a subset of R^3, for example; then a function

$$\begin{aligned} \phi : D &\longrightarrow R \\ (x, y, z) &\longmapsto u \end{aligned} \qquad \text{or} \qquad \begin{aligned} \phi : D \times R_0^+ &\longrightarrow R \\ (x, y, z, t) &\longmapsto u \end{aligned}$$

is called a **scalar field**. Thus a scalar field is just a generalization of the type of function encountered in ordinary differential equations, i.e. with domain and codomain a subset of the reals. We define addition of scalar fields pointwise over the domain as follows. If ϕ, ψ are two scalar fields with domain $D \subset R^3$ then $\phi + \psi$ is the scalar field defined by

$$(\phi + \psi)(x, y, z) = \phi(x, y, z) + \psi(x, y, z).$$

Similarly we may define scalar multiplication by

$$(m\phi)(x, y, z) = m(\phi(x, y, z)) \qquad m \in R.$$

With respect to these definitions the set of all continuous scalar fields with a given domain form a vector space.

A **vector field** is a function $\mathbf{v}$ with domain D (in R^2) or $D \times R_0^+$ and codomain G^2 (the vector space of geometric vectors in the plane). Thus

$$\mathbf{v} : (x, y) \longmapsto v_x(x, y)\mathbf{i} + v_y(x, y)\mathbf{j}$$

where v_x and v_y are scalar fields with the same domain as $\mathbf{v}$. We can also define three-dimensional vector fields $R^3 \longrightarrow G^3$.

By defining addition and scalar multiplication of vector fields as for scalar fields we again obtain a vector space of vector fields on a given domain.

The temperature in a block of metal is an example of a scalar field, and the local velocity in a liquid determines a vector field.

The **gradient** of a scalar field ϕ is the vector field

$$\mathbf{grad}\ \phi = \frac{\partial \phi}{\partial x}\mathbf{i} + \frac{\partial \phi}{\partial y}\mathbf{j} + \frac{\partial \phi}{\partial z}\mathbf{k}.$$

The **divergence** of a vector field $\mathbf{v} = (v_x, v_y, v_z)$ is the scalar field

$$\operatorname{div} \mathbf{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}.$$

Note that

$$\operatorname{div} \mathbf{grad}\, \phi = \nabla^2 \phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2}.$$

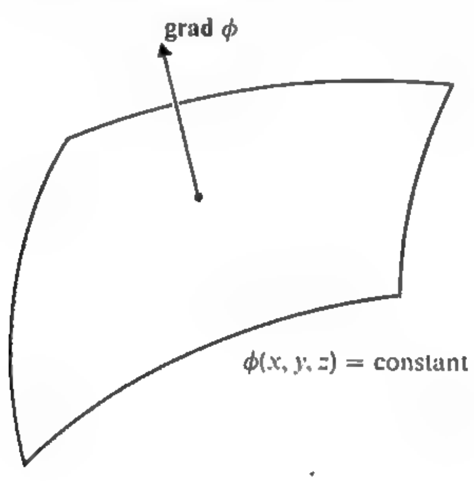
If S, V denote the vector spaces of scalar and vector fields respectively on some given domain, then $\mathbf{grad}: S \longrightarrow V$ and $\operatorname{div}: V \longrightarrow S$ are *linear* operators (vector space morphisms). We have written **grad** for the gradient operator to indicate that **grad** ϕ is a *vector* field; Weinberger, however, writes grad. We can define two-dimensional versions of **grad**, div and ∇^2 by omitting the z-dependence. You should note that $\partial/\partial t$ does not figure in these operators even for time-dependent fields.

Our definitions of **grad** and div presuppose a rectangular Cartesian coordinate system. Definitions which avoid this presupposition are possible (see *Unit MST 282 7, Work and Energy I*, Section 7.1). However the formal definition given above is sufficient for our purposes.

The **directional derivative** of the scalar field ϕ at (x, y, z) in the direction of the unit vector $\mathbf{e}$ is defined to be the real number

$$\mathbf{grad}\, \phi \cdot \mathbf{e},$$

which is often denoted by $\partial\phi/\partial e$. It measures the rate of change of ϕ in the direction of $\mathbf{e}$. Since $\mathbf{e} \cdot \mathbf{grad}\, \phi = |\mathbf{grad}\, \phi| \cos \theta$, where θ is the angle between $\mathbf{e}$ and $\mathbf{grad}\, \phi$, we see that the greatest rate of change of ϕ occurs when $\mathbf{e}$ is parallel to $\mathbf{grad}\, \phi$.



The solution set of the equation

$$\phi(x, y, z) = k$$

represents a surface in R^3 for each value of the constant k. If the unit vector $\mathbf{e}$ is tangential to the surface we expect $\mathbf{e} \cdot \mathbf{grad}\, \phi = 0$, that is, the rate of change $\partial\phi/\partial e$ is zero since ϕ is constant on the surface. Since this is true for any tangent it follows that **grad** ϕ is *normal* (i.e. perpendicular) to the surface at the point. As we might expect, the greatest rate of change of ϕ occurs in the direction normal to the surface.

For two-dimensional problems the solution set of

$$\phi(x, y) = k$$

represents a curve in R^2, and the previous discussion is equally valid.

In general we call any vector $\mathbf{t}$ such $\mathbf{t} \cdot \mathbf{grad}\, \phi = 0$ at the point P a **tangent vector** to the constant value surface (or curve) of ϕ passing through P. The set of all such t forms the **tangent plane** (or **line**) to the surface (or curve) at P. The unit normal vector at P to the surface (or curve) is clearly given by $(\mathbf{grad}\, \phi)/|\mathbf{grad}\, \phi|$ evaluated at P.

SAQ 2

Find the gradients of the following scalar fields.

(a) $(x, y, z) \longmapsto (x^2 + y^2 + z^2)^{\frac{1}{2}}$

(b) $(x, y, z) \longmapsto xyz$

(c) $(x, y, z) \longmapsto x^2 + y^2 z + z^3$

(Solution on p. 34.)

SAQ 3

If $\mathbf{F}:(x, y, z) \longmapsto (2xy + z^3)\mathbf{i} + x^2\mathbf{j} + 3xz^2\mathbf{k}$, find div $\mathbf{F}$.

(Solution on p. 34.)

SAQ 4

Find the rate of change of $\phi:(x, y, z) \longmapsto xyz$ at the point $(1, 2, 1)$ in the direction of $\mathbf{i} - \mathbf{j} - \mathbf{k}$.

(Solution on p. 34.)

READ W: Section 10, page 48 to page 50, two lines below Equation (10.2).

Notes

(i) *W: page 48, lines 1 to −10*
The membrane may be thought of as a thin sheet of rubber stretched over a frame as in the case of a drum, or as a soap film on a metal frame (see cover). Ideally the material is perfectly flexible in that it does not resist bending but does resist stretching.

The membrane equation quoted on *line −10* is the natural analogue in two dimensions of the equation for the vibrating string derived in *Unit 1, The Wave Equation*. In fact, if c, F and u are independent of y, then this equation (*line −10*) reduces to the one-dimensional vibrating string equation. The detailed derivation of the membrane equation is complicated and requires consideration of the curvature of a surface. An elementary treatment can be found on page 43 of Coulson, *Waves*.

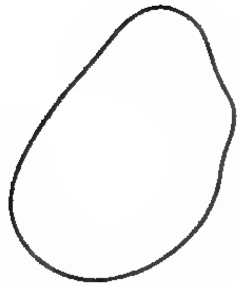
(ii) *W: page 49, line 7*
The second condition is equivalent to

$$\frac{y'(\tau_1)}{x'(\tau_1)} = \frac{y'(\tau_0)}{x'(\tau_0)},$$

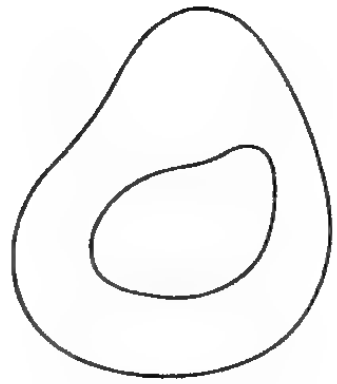
if $x'(\tau_1) \neq 0$ and $x'(\tau_0) \neq 0$. This ensures that the curve C is smooth at its common point $(x(\tau_0), y(\tau_0)) = (x(\tau_1), y(\tau_1))$.

(iii) *W: page 49, footnote*
For example the interior of the circle $x^2 + y^2 = 1$ is a simply connected domain. On the other hand the domain bounded by the circles $x^2 + y^2 = 2$ and $x^2 + y^2 = 1$ is not simply connected. A domain in R is just an open interval.

A domain is not necessarily bounded. For example, the subset $\{(x, y): x > 0, y > 0\}$ of R^2 is a domain.



simply connected



not simply connected

(iv) *W: page 49, line −12 to page 50, line 17*
A knowledge of electromagnetic theory is not required for this course. If you find the next passage difficult in that respect, read through it and extract just the mathematics.

(v) *W: page 50, line 15*
In this two-dimensional case, the tangent vector

$$\mathbf{t} = \frac{dx}{d\tau}\mathbf{i} + \frac{dy}{d\tau}\mathbf{j}$$

to C and the gradient vector

$$\mathbf{grad}\, u = \frac{\partial u}{\partial x}\mathbf{i} + \frac{\partial u}{\partial y}\mathbf{j}$$

lie in the same line. Thus $\mathbf{t} \times \mathbf{grad}\, u = \mathbf{0}$ produces the required equation.

General Comment

A boundary value problem in which u satisfies Laplace's equation in D, and is specified on C, is known as a **Dirichlet problem**; if, alternatively, $\partial u/\partial n$ is specified on C, then it is known as a **Neumann problem**. Sometimes a mixed problem may be given, in which u is specified on part of C and $\partial u/\partial n$ on the remainder.

A **solution to the problem** is then a function u which is continuous on $D \cup C$ and satisfies the equation and the boundary conditions.

SAQ 5

Show that the membrane equation in Cartesian coordinates,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = -F(x, y),$$

transforms into

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = -G(r, \theta)$$

in polar coordinates (r, θ) where $x = r\cos\theta$, $y = r\sin\theta$ and $G(r, \theta) = F(r\cos\theta, r\sin\theta)$.

If $G(r, \theta) = r$ inside the circle $C: x^2 + y^2 = a^2$ and $u = 0$ on C, find a solution which is bounded and has radial symmetry (i.e. depends on r alone).

(Solution on p. 34.)

SAQ 6

Show that

$$\Psi(r, \theta) = b\left(r - \frac{a^2}{r}\right)\sin\theta,$$

where (r, θ) are polar coordinates and a, b are constants, satisfies Laplace's equation $\nabla^2\Psi = 0$ for all $r > 0$. (HINT: Use the form of the Laplace operator obtained in SAQ 5.)

Sketch the curves $\Psi(r, \theta) = c$ for various values of the constant c in the domain $r > a$. What do these curves look like for large r?

In the context of fluid dynamics, Ψ is called a *stream function* and the curves along which the value of Ψ is constant are the *streamlines* for incompressible, non-viscous flow. The *fluid velocity* is the vector field represented in rectangular Cartesian coordinates by

$$(x, y) \longmapsto \left(\frac{\partial\Psi}{\partial y}, -\frac{\partial\Psi}{\partial x}\right).$$

Can you guess what the form for Ψ in this problem represents physically?

(Solution on p. 35.)

3.1.2 Laplace's Equation and Boundary Data

READ W: page 50, line −6 to page 52, the end of Section 10.

General Comment

In Hadamard's example small changes in boundary data lead to large changes in the interior solution since

$$\lim_{n\to\infty} u_n(\tfrac{1}{2}, y)$$

does not exist for any fixed $y \neq 0$. Since $\lim f(x) = 0$ for large n, and $u(x, y) = 0$ is a solution of the problem in which f is the zero function, we might expect the limit of $u_n(x, y)$ for $x \in (0, 1)$, $y > 0$ to be zero too, which is clearly not the case. This is what we mean by saying that the solution is not continuous with respect to its data.

A boundary or initial value problem for a partial differential equation is said to be **properly posed** if and only if its solution *exists*, is *unique* and depends *continuously* on the assigned data. Generally speaking, a problem which is not properly posed is unlikely to represent a physical problem.

SAQ 7

Show that

$$e^{-\sqrt{n}}\cosh(4n + 1)\pi y \qquad n \in Z^+$$

is unbounded for any fixed $y \neq 0$.

(Solution on p. 36.)

SAQ 8

W: page 52, Exercise 3.

(Solution on p. 36.)

3.1.3 Green's Theorem

Before you read the next passage from *W* we need to remind you of, or introduce you to *surface integrals* and *line integrals*. The former are described in *Unit MST 282 6, Rigid Bodies*, Section 6.2.3 and the latter in *Unit MST 282 7*, Section 7.1.1; alternatively, you could refer to Spiegel, *Vector Analysis*, Chapter 5. The essentials are summarized below.

The **surface integral** $\iint_D f\,dA$ of f over the surface D represents the limit of a sum formed by adding together the products $f_P \Delta A$ where f_P is the value of f at a point P in the small element ΔA of the surface D. When the surface lies in a plane the integral may be evaluated by introducing coordinates so that it can be converted into an equivalent *double integral*. In a rectangular Cartesian frame of coordinates Oxy, we have $\Delta A = \Delta x \Delta y$ and the integral is evaluated as

$$\iint_D f(x, y)dx\,dy.$$

In polar coordinates (r, θ), we have $\Delta A = r\Delta r \Delta\theta$ and the integral becomes

$$\iint_D f(r\cos\theta, r\sin\theta)r\,dr\,d\theta.$$

(In a proper treatment we should check that these two integrals, are, in fact, equal.)

To interpret the double integral we must imagine the inner integral $\left(\int f(x, y)dx, \text{ say}\right)$ to be enclosed in brackets. The limits of integration must be suitably determined and it is not always a simple matter to obtain them for a given domain D.

The order of integration, that is, whether to evaluate

$$\iint_D f(x, y)dx\,dy$$

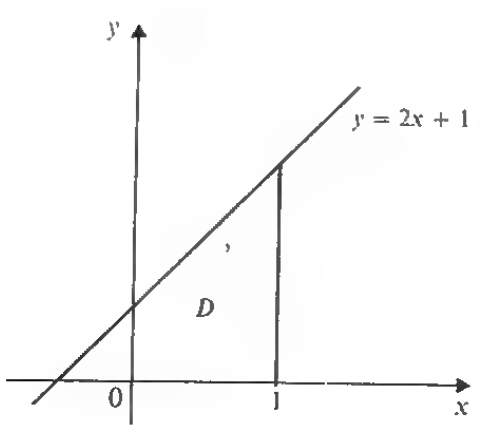
or

$$\iint_D f(x, y)dy\,dx$$

does not normally matter, but a judicious choice will simplify the limits of integration.

Example

Evaluate $\iint_D f dA$ where $f(x, y) = xy$ and D is the area shaded in the diagram.



Solution

In this case

$$\iint_D xy\,dA = \int_0^1 \left\{ \int_0^{2x+1} xy\,dy \right\} dx.$$

(The braces are not normally used; we have incorporated them here to remove any doubt as to which integral sign refers to which integration.)

$$\int_0^1 \left\{ \int_0^{2x+1} xy\, dy \right\} dx = \int_0^1 x \left[\frac{y^2}{2} \right]_0^{2x+1} dx$$
$$= \int_0^1 \frac{x(2x+1)^2}{2}\, dx$$
$$= \frac{1}{2}\left[x^4 + \frac{4x^3}{3} + \frac{x^2}{2} \right]_0^1 = \frac{17}{12}.$$

(It is not essential to integrate over y first— we could integrate over x first. In this case it would be more complicated to do so.)

SAQ 9

Evaluate

$$\iint_D \operatorname{div} \mathbf{v}\, dA$$

where D is the disc

$$\{(x, y): x^2 + y^2 < 1\}$$

and

$$\mathbf{v} = x\sqrt{(1 - x^2)}\mathbf{i} + y\sqrt{(1 - x^2)}\mathbf{j}.$$

(Solution on p. 37.)

The **line integral** $\int_C f\, ds$ represents the limit of a sum formed by adding together the products of $f_P \Delta s$ where f_P is the value of f at a point P in the small element of length Δs of the curve C. We use the notation

$$\oint_C$$

to indicate that the curve C is closed, and unless otherwise stated, that C is described counter-clockwise. If the curve is given by

$$\phi: \tau \longmapsto (x, y) \qquad \tau \in [\tau_0, \tau_1]$$

then the arc length is given by

$$s(\tau) = \int_{\tau_0}^{\tau} \left\{ \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 \right\}^{\frac{1}{2}} dt,$$

so that

$$\frac{ds}{d\tau} = \left\{ \left(\frac{dx}{d\tau}\right)^2 + \left(\frac{dy}{d\tau}\right)^2 \right\}^{\frac{1}{2}}.$$

For example, if ϕ is given in terms of x we obtain the identity

$$\frac{ds}{dx} = \left\{ 1 + \left(\frac{dy}{dx}\right)^2 \right\}^{\frac{1}{2}}.$$

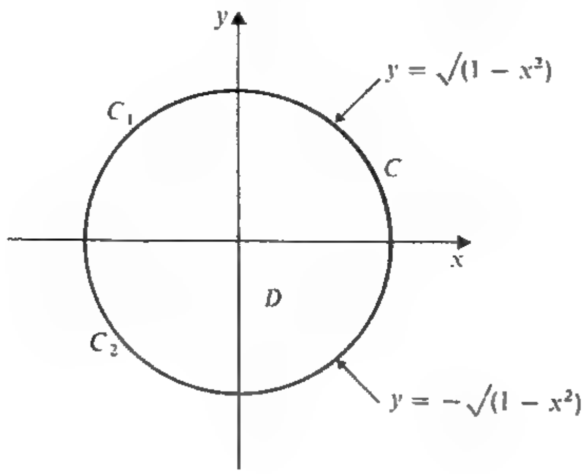
The line integral $\int_C f\, ds$ then becomes the definite integral

$$\int f \left\{ 1 + \left(\frac{dy}{dx}\right)^2 \right\}^{\frac{1}{2}} dx$$

between appropriate limits.

Example

Evaluate $\oint_C \mathbf{v}\cdot\mathbf{n}\,ds$ where C is the circle $x^2+y^2=1$ and $\mathbf{v}=x\sqrt{(1-x^2)}\mathbf{i}+y\sqrt{(1-x^2)}\mathbf{j}$, and $\mathbf{n}$ is the unit outward normal to the circle at the point (x, y).



Solution

The unit outward normal $\mathbf{n}$ is given by

$$\mathbf{n}=\frac{\mathbf{grad}\,(x^2+y^2)}{|\mathbf{grad}\,(x^2+y^2)|}$$

(see Section 3.1.1). Thus

$$\mathbf{n}=\frac{x\mathbf{i}+y\mathbf{j}}{\sqrt{(x^2+y^2)}}$$
$$=x\mathbf{i}+y\mathbf{j}$$

since $x^2+y^2=1$ on C. Hence,

$$\mathbf{v}\cdot\mathbf{n}=x^2\sqrt{(1-x^2)}+y^2\sqrt{(1-x^2)}=\sqrt{(1-x^2)}$$

on C. Let C_1 be the upper part of C, on which

$$y=\sqrt{(1-x^2)}$$

and C_2 the lower part, on which

$$y=-\sqrt{(1-x^2)}.$$

On C_1 we have

$$\frac{dy}{dx}=-\frac{x}{\sqrt{(1-x^2)}},$$

so that

$$\frac{ds}{dx}=\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{\frac{1}{2}}$$
$$=\left(1+\frac{x^2}{1-x^2}\right)^{\frac{1}{2}}$$
$$=-\frac{1}{\sqrt{(1-x^2)}},$$

taking the negative square root since we are traversing C counter-clockwise, and so ds/dx is negative on C_1. Thus

$$\int_{C_1}\mathbf{v}\cdot\mathbf{n}\,ds=\int_1^{-1}\sqrt{(1-x^2)}\cdot\frac{-1}{\sqrt{(1-x^2)}}\,dx=\int_{-1}^{1}dx=2.$$

Similarly, on C_2 we have

$$\frac{ds}{dx} = \frac{1}{\sqrt{(1 - x^2)}}.$$

Thus

$$\int_{C_2} \mathbf{v} \cdot \mathbf{n}\, ds = \int_{-1}^{1} \sqrt{(1 - x^2)} \cdot \frac{1}{\sqrt{(1 - x^2)}}\, dx = \int_{-1}^{1} dx = 2.$$

Finally

$$\oint_C \mathbf{v} \cdot \mathbf{n}\, ds = \int_{C_1} \mathbf{v} \cdot \mathbf{n}\, ds + \int_{C_2} \mathbf{v} \cdot \mathbf{n}\, ds = 4.$$

It is not by chance that this numerical result is the same as that for SAQ 9; it is a particular example of the Divergence Theorem which you will meet shortly.

READ W: Section 11, page 52 to page 53, two lines below Equation (11.2).

Notes

(i) *W: page 52, Equation (11.1)*
Here, div, **grad** and ∇^2 are to be interpreted as operators on functions of two variables. An analogous result holds when the operators act on functions of three variables.

(ii) *W: page 53, lines 5 to 8*
A proof of the Divergence Theorem is given following this reading passage.

The name *Stokes' Theorem* is usually reserved for a particular generalization of this result to higher dimensions.

DIVERGENCE THEOREM (IN TWO DIMENSIONS)

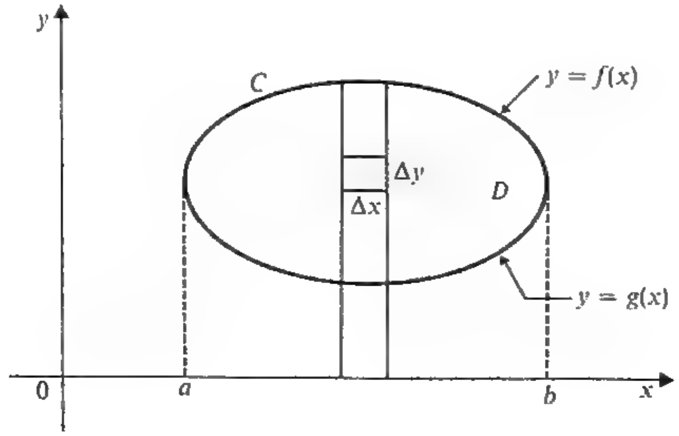
Let D be a convex domain in the plane bounded by the closed curve C, and let $\mathbf{v} = (v_x, v_y)$ be a vector field, where $(x, y) \longmapsto v_x$ and $(x, y) \longmapsto v_y$ are continuously differentiable in D and continuous in $D \cup C$. Then

$$\iint_D \operatorname{div} \mathbf{v}\, dA = \oint_C \mathbf{v} \cdot \mathbf{n}\, ds,$$

where $\mathbf{n}$ is the outward unit normal on C. (A domain in R^n is said to be **convex** if no straight line intersects its boundary in more than two points.)

Proof

Assume that C is bounded by the lines $x = a$ and $x = b$ $(a < b)$, as shown. Let $y = f(x)$ on the upper part of C, and $y = g(x)$ on the lower part.



Then

$$\iint_D \frac{\partial v_y}{\partial y}\,dA = \int_a^b \left\{\int_{g(x)}^{f(x)} \frac{\partial v_y}{\partial y}\,dy\right\} dx$$

$$= \int_a^b \left(v_y|_{y=f(x)} - v_y|_{y=g(x)}\right) dx$$

$$= -\int_a^b v_y|_{y=g(x)}dx - \int_b^a v_y|_{y=f(x)}dx$$

$$= -\oint_C v_y dx.$$

A similar argument in which the integration with respect to x is performed first gives

$$\iint_D \frac{\partial v_x}{\partial x}\,dA = \oint_C v_x dy,$$

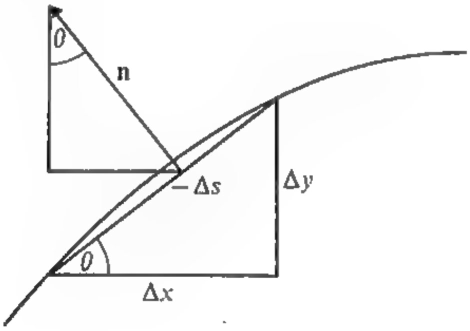
but note that a different partition of the curve C will be required. Addition gives

$$\iint_D \operatorname{div} \mathbf{v}\,dA = \oint_C v_x dy - v_y dx.$$

To show that

$$\oint \mathbf{v}\cdot\mathbf{n}\,ds = \int_C v_x dy - v_y dx,$$

we note from the figure that $\Delta x \doteq -\Delta s\cos\theta$, $\Delta y = -\Delta s\sin\theta$. (Note that C is traversed counter-clockwise.)



The unit outward normal is

$$\mathbf{n} = -\sin\theta\,\mathbf{i} + \cos\theta\,\mathbf{j}$$

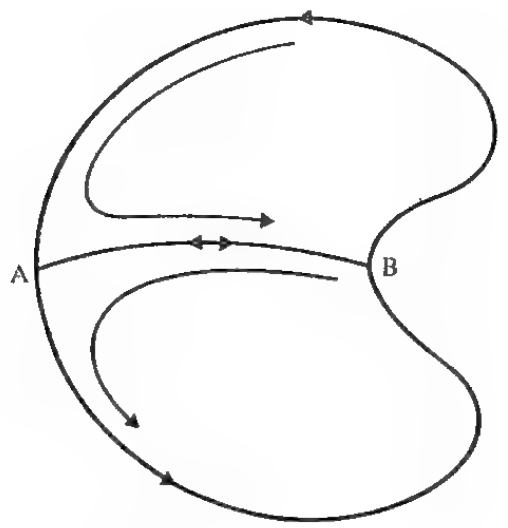
$$= \frac{dy}{ds}\mathbf{i} - \frac{dx}{ds}\mathbf{j},$$

so that

$$\oint_C v_x dy - v_y dx = \oint_C \mathbf{v}\cdot\mathbf{n}\,ds$$

If a domain is not convex, but can be divided into a finite number of convex sub-domains, then the theorem holds for each sub-domain, and for the whole domain by

addition, since it turns out that the line integrals along internal boundaries cancel in pairs. (In the figure the contributions along AB and BA balance each other.)



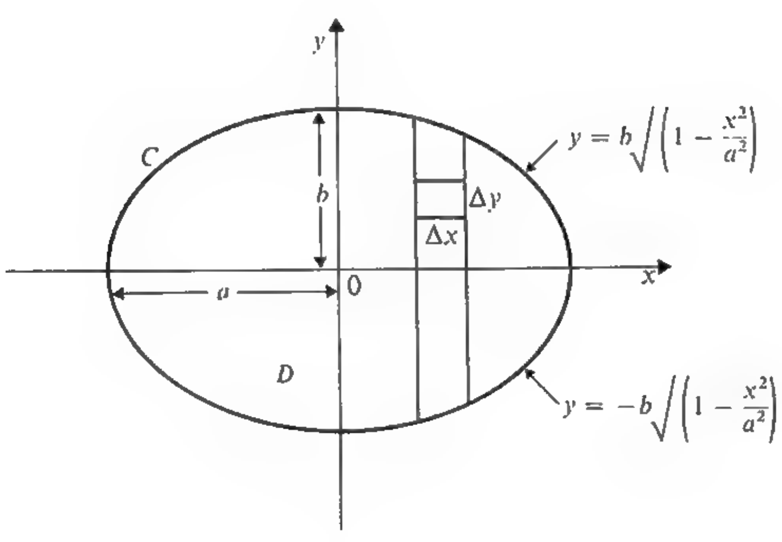
Example

Find the area enclosed by the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

using a line integral.

Solution



The area A enclosed by the ellipse is defined by

$$A = \iint_D dA.$$

(You may verify that this definition of area is equivalent to that in *Unit M100 9, Integration I*.) Now put $\mathbf{v} = (x, y)$ and apply the Divergence Theorem. Since

$$\operatorname{div} \mathbf{v} = 2,$$

we obtain

$$A = \frac{1}{2}\oint_C (xdy - ydx).$$

We can represent the ellipse by

$$t \longmapsto (a \cos t, b \sin t) \qquad t \in [0, 2\pi],$$

the description being counter-clockwise as t increases. (Check that this satisfies the equation for the ellipse.) Thus, by the rule for substitution,

$$\begin{aligned} A &= \frac{1}{2}\int_0^{2\pi}\left(x\frac{dy}{dt} - y\frac{dx}{dt}\right)dt \\ &= \int_0^{2\pi} (a\cos t \,.\, b\cos t + b\sin t \,.\, a\sin t)dt \\ &= \tfrac{1}{2}ab\int_0^{2\pi} dt = \pi ab. \end{aligned}$$

The area obtained agrees (unsurprisingly) with the result in *Unit M100 13, Integration II*, Section 13.2.5.

SAQ 10

Evaluate $\oint_C \mathbf{v}\cdot\mathbf{n}\,ds$ where $\mathbf{v} = (-5x, -2y^2)$ and C is the triangle whose vertices are (0, 0), (1, 0), (0, 1). Verify that it equals $\iint_D \operatorname{div}\mathbf{v}\,dA$ where D is the interior of the triangle.

(Solution on p. 37.)

SAQ 11

Establish the following identities:

(a) $\operatorname{div}(a\,\mathbf{b}) = a \operatorname{div}\mathbf{b} + \mathbf{grad}\, a\cdot\mathbf{b}$;

(b) $\displaystyle\iint_D u\nabla^2 v\,dA = \oint_C u\frac{\partial v}{\partial n}ds - \iint_D \mathbf{grad}\,u\cdot\mathbf{grad}\,v\,dA$;

(c) $\displaystyle\iint_D (u\nabla^2 v - v\nabla^2 u)dA = \oint_C\left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\right)ds$;

where a, u and v are scalar fields and $\mathbf{b}$ is a vector field, and C is the boundary of the convex domain D.

(Solution on p. 38.)

3.1.4 Uniqueness

READ W: page 53, line −14 to page 54 (the end of Section 11).

Notes

(i) *W: page 53, lines −7 and −6*
u_1 and u_2 both satisfy Equations (10.1) on *W: page 48*; that is

$$\nabla^2 u_i = -F \quad \text{in } D,$$
$$u_i = f \quad \text{on } C,$$

for $i = 1, 2$. By linearity, the difference $v = u_1 - u_2$ satisfies

$$\nabla^2 v = 0 \quad \text{in } D,$$
$$v = 0 \quad \text{on } C.$$

(ii) *W: page 54, line 1*
This is a *mixed* boundary value problem in which values of u and $\partial u/\partial n$ are given over different parts of the boundary.

SAQ 12

Show that the boundary value problem

$$\nabla^2 u = 0 \quad \text{in } D,$$

$$\frac{\partial u}{\partial n}(x, y) + h(x, y)u(x, y) = g(x, y) \quad \text{and} \quad h(x, y) > 0 \quad \text{on } C,$$

where C is a piecewise continuously differentiable curve bounding the domain D, has at most one solution which is continuous on $D \cup C$.

(Solution on p. 39.)

3.2 EXTREMUM PRINCIPLES

3.2.0 Introduction

It is often impossible to find analytical forms for solutions to Poisson's (or Laplace's) equation even when the data are specified on a boundary curve with relatively simple geometry. Numerical techniques are required and these are discussed in detail later in this course.

In this section we develop some general inequalities which provide bounds on the value of the solution u in the domain D on whose boundary data are given, and on the integral

$$\iint_D u \, dA.$$

We have set the discussion in the context of fluid mechanics, with particular reference to steady flow of a viscous fluid in a pipe, although the techniques can also be used with success in analogous problems such as the twisting of elastic bars. Generally speaking the method will not give exact solutions, or even good local estimates of solutions, but it remains an approach which often provides results when most other forms of non-numerical analysis fail.

3.2.1 The Maximum Principle

READ W: Section 12, page 55 to page 56, line 12.

Notes

(i) *W: page 55, line 2*
Although the Law of Conservation of Energy was used to prove the uniqueness property, we derived it mathematically (*W: page 53*) from the problem, and not from a consideration of the physics.

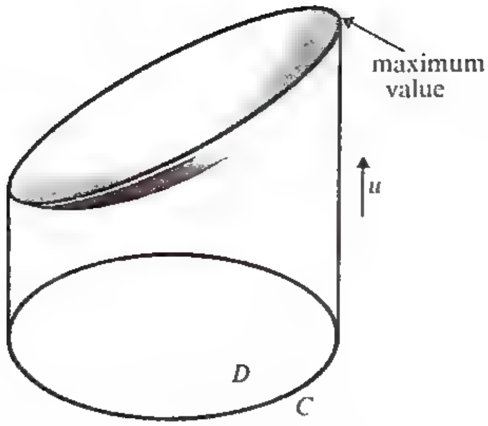
(ii) *W: page 55, line 8*
u attains its maximum on $D \cup C$ (denoted in W by $D + C$), as a result of a theorem in analysis which says that if a function, with domain a closed, bounded subset of R^n and codomain R (the reals), is continuous, then its values are bounded and it attains its bounds.

(iii) *W: page 55, lines 9 to 14*
Clearly, since u has a maximum at $(x_0, y_0) \in D$, the function $x \longmapsto u|_{y=y_0}$ has a maximum at x_0, and the function $y \longmapsto u|_{x=x_0}$ has a maximum at y_0. Now, we know that at a point where a function of a single variable has a stationary value its derivative is zero, and for a maximum the second derivative is non-positive (*Unit M100 15, Differentiation II*). Hence,

$$\left.\frac{\partial u}{\partial x}\right|_{(x_0,y_0)} = 0, \quad \left.\frac{\partial^2 u}{\partial x^2}\right|_{(x_0,y_0)} \leqslant 0,$$

and

$$\left.\frac{\partial u}{\partial y}\right|_{(x_0,y_0)} = 0, \quad \left.\frac{\partial^2 u}{\partial y^2}\right|_{(x_0,y_0)} \leqslant 0.$$



On C, however, it is possible to have a maximum which is not a stationary value.

(iv) *W: page 55, line 18*
Note that the foregoing proof must fail if F takes the value zero at some point in D since

$$\nabla^2 u|_{(x_0, y_0)} \leqslant 0$$

and

$$F(x_0, y_0) \leqslant 0$$

are not necessarily contradictory. Thus a more elaborate proof is required for the case $F(x, y) \leqslant 0$.

General Comment

Although we have not asked you to study the proof of continuity with respect to the data for the Dirichlet Problem, you should be familiar with the result.

SAQ 13

Two harmonic functions $(x, y) \longmapsto u_1$ and $(x, y) \longmapsto u_2$ satisfy $u_1 < u_2$ on a piecewise differentiable closed curve C. Show that $u_1 < u_2$ throughout the domain D bounded by C.

(Solution on p. 39.)

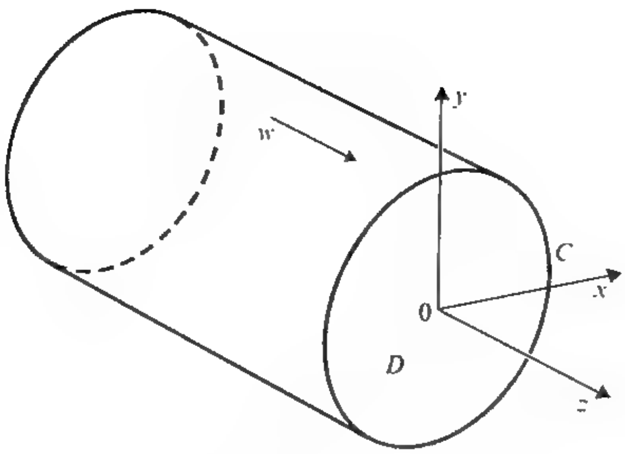
3.2.2 Flow in a Pipe

Consider the steady flow of an incompressible viscous fluid along a straight pipe of uniform cross-section. By steady flow we mean that the speed of the flow w is independent of time at each point. Incompressibility implies that w must also be independent of z, the distance along the pipe (as shown in the figure), for otherwise a given mass of liquid would be compressed into a smaller volume or expanded into a larger volume. If D is a section of the fluid in the pipe then it can be shown that w satisfies

$$\nabla^2 w = -k \quad \text{in } D. \tag{1}$$

Here $k = p/\rho\nu$ is a positive constant; p is the (constant and) uniform pressure gradient along the pipe, ρ is the density of the fluid and ν is its kinematic viscosity, a measure of the internal friction of the fluid. A viscous fluid adheres to the boundary (think of oil flowing down a metal plate, or why the engine oil level can be measured with a dipstick) so that

$$w = 0 \quad \text{on } C. \tag{2}$$



This problem is equivalent to that of a membrane, attached to a plane frame, with one side subjected to a uniform pressure (*W: page 48*).

By the maximum principle of Section 3.2.1, w must attain its *minimum* on C, since $\nabla^2(-w) = k > 0$. Thus $w \geqslant 0$ in D; in other words, local flow against the pressure gradient is impossible.

The **total flux**, defined by

$$Q = \iint_D w\,dA, \tag{3}$$

is of physical interest since it measures the rate of flow through the pipe. Since $1 = -(\nabla^2 w)/k$, we can express Equation (3) as

$$\begin{aligned} Q &= -\frac{1}{k}\iint_D w\nabla^2 w\,dA, \\ &= -\frac{1}{k}\oint_C w\frac{\partial w}{\partial n}\,ds + \frac{1}{k}\iint_D |\textbf{grad}\ w|^2\,dA, \end{aligned}$$

using Green's Theorem (*W: page 53*). However, the first integral on the right vanishes by Equation (2). Thus

$$Q = \frac{1}{k}\iint_D |\textbf{grad}\ w|^2\,dA.$$

SAQ 14

Verify that the fluid velocity w in a circular pipe of radius a is given by

$$w = \tfrac{1}{4}k(a^2 - x^2 - y^2),$$

where k is as defined in the text. Find the flux through the pipe.

(Solution on p. 39.)

SAQ 15

Find an expression for the speed of flow in a pipe with an elliptic section given by

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

when k is as given in the text.

HINT: Guess the solution by generalizing the result for a pipe with a circular section, and then verify.

Show that the flux is given by

$$Q = \frac{\pi k a^3 b^3}{4(a^2 + b^2)}.$$

3.2.3 Extremum Principles for Viscous Flow

In order to derive bounds on the flux for the viscous flow problem, we first obtain a result about vector fields on a domain D in R^2, that is, continuous functions $D \longrightarrow G^2$, where G^2 is the set of geometric vectors in the plane.

We have seen in Section 3.1.1 that the set of vector fields on D forms a vector space with respect to pointwise addition and scalar multiplication of vector fields.

For two vector fields

$$\mathbf{a}: D \longrightarrow G^2, \qquad \mathbf{b}: D \longrightarrow G^2,$$

let $\mathbf{a} \cdot \mathbf{b}: D \longrightarrow R$ denote the scalar field whose value at each point of D is given by the usual inner product for geometric vectors. Thus $\mathbf{a} \cdot \mathbf{b}$ is the mapping

$$(x, y) \longmapsto \mathbf{a}(x, y) \cdot \mathbf{b}(x, y) \qquad (x, y) \in D.$$

Now, define the binary operation $\square$ on the space of vector fields by the surface integral (expressed in function notation)

$$\mathbf{a} \square \mathbf{b} = \iint_D \mathbf{a} \cdot \mathbf{b}.$$

It is not difficult to see that for any two vector fields $\mathbf{a}, \mathbf{b}$ on D the operation $\square$ yields a real number and satisfies the axioms for an *inner product* (you need not verify this statement). With respect to this inner product, *Schwarz's inequality* (see Appendix) becomes

$$\left(\iint_D \mathbf{a} \cdot \mathbf{b}\right)^2 \leqslant \left(\iint_D a^2\right)\left(\iint_D b^2\right).$$

We shall employ this result to put bounds on the flux Q, which, as we have seen in the previous section, can be expressed as

$$\frac{1}{k}\iint_D |\mathbf{grad}\, w|^2 \, dA \tag{4}$$

for viscous flow in a uniform pipe.

FIRST EXTREMUM PRINCIPLE

Let w satisfy $\nabla^2 w = -k$ in a domain D and $w = 0$ on its boundary C. Then the total flux

$$Q = \iint_D w \, dA$$

is bounded below by

$$k\left(\iint_D w^* \, dA\right)^2 \Big/ \iint_D |\mathbf{grad}\, w^*|^2 \, dA,$$

where w^* is any non-zero scalar field on $D \cup C$ with the property that $w^* = 0$ on C.

Proof

If we set $u = w^*$, $v = w$ in the identity proved in part (b) of SAQ 11, we obtain

$$\iint_D w^*\nabla^2 w\, dA = \oint_C w^* \frac{\partial w}{\partial n}\, ds - \iint_D \mathbf{grad}\, w^* \cdot \mathbf{grad}\, w\, dA.$$

Since $w^* = 0$ on C and $\nabla^2 w = -k$ in D, this reduces to

$$k \iint_D w^*\, dA = \iint_D \mathbf{grad}\, w^* \cdot \mathbf{grad}\, w\, dA.$$

We now apply Schwarz's inequality with $\mathbf{a} = \mathbf{grad}\, w^*$ and $\mathbf{b} = \mathbf{grad}\, w$, to obtain

$$k^2\left(\iint_D w^*\, dA\right)^2 \leqslant \iint_D |\mathbf{grad}\, w|^2\, dA \iint_D |\mathbf{grad}\, w^*|^2\, dA$$

$$\leqslant kQ \iint_D |\mathbf{grad}\, w^*|^2\, dA \qquad \text{using expression (4) for } Q.$$

The result follows since $k > 0$, and so

$$Q \geqslant k\left(\iint_D w^*\, dA\right)^2 \Big/ \iint_D |\mathbf{grad}\, w^*|^2\, dA.$$

SECOND EXTREMUM PRINCIPLE

Let w satisfy $\nabla^2 w = -k$ in a domain D and $w = 0$ on its boundary C. Then the total flux

$$Q = \iint_D w\, dA$$

is bounded above by

$$\frac{1}{k}\iint_D v^{*2}\, dA,$$

where $\mathbf{v}^*$ is any vector field which has the property that $\operatorname{div} \mathbf{v}^* = -k$ in D.

Proof

Since $\operatorname{div}(w\mathbf{v}^*) = w \operatorname{div} \mathbf{v}^* + \mathbf{v}^* \cdot \mathbf{grad}\, w$ at all points of D, by part (a) of SAQ 11,

$$\iint_D \mathbf{v}^* \cdot \mathbf{grad}\, w\, dA = \iint_D \operatorname{div}(w\mathbf{v}^*)\, dA - \iint_D w \operatorname{div} \mathbf{v}^*\, dA$$

$$= \oint_C w\mathbf{v}^* \cdot \mathbf{n}\, ds - \iint_D w \operatorname{div} \mathbf{v}^*\, dA,$$

using the Divergence Theorem (Section 3.1.3). Since $w = 0$ on C and $\operatorname{div} \mathbf{v}^* = -k$ in D, this reduces to

$$\iint_D \mathbf{v}^* \cdot \mathbf{grad}\, w\, dA = kQ,$$

by the definition of Q. Applying Schwarz's inequality with $\mathbf{a} = \mathbf{v}^*$ and $\mathbf{b} = \mathbf{grad}\, w$, we obtain

$$k^2Q^2 \leqslant \iint_D |\mathbf{grad}\, w|^2\, dA \iint_D v^{*2}\, dA$$

$$\leqslant kQ \iint_D v^{*2}\, dA$$

using expression (4) for Q.

The result follows since $k > 0$.

Fields w^* and $\mathbf{v}^*$ which satisfy the conditions above are said to be admissible functions for the First and Second Extremum Principle, respectively. We have established that, for admissible functions w^* and $\mathbf{v}^*$,

$$\frac{k\left(\iint_D w^*\,dA\right)^2}{\iint_D |\mathbf{grad}\, w^*|^2\,dA} \leqslant Q \leqslant \frac{1}{k}\iint_D v^{*2}\,dA.$$

Example

Find bounds for the flux in a pipe of square section, $(x, y) \in [-a, a] \times [-a, a]$.

Solution

To obtain the upper bound we choose $\mathbf{v}^* = (-\frac{1}{2}kx, -\frac{1}{2}ky)$ which is admissible since it satisfies div $\mathbf{v}^* = -k$. (This could be used for a pipe of any cross-section.) Thus, by the Second Extremum Principle,

$$Q \leqslant \tfrac{1}{4}k\int_{-a}^{a}\int_{-a}^{a}(x^2 + y^2)\,dx\,dy = \tfrac{2}{3}ka^4.$$

For the lower bound we require a function which vanishes on C; we choose

$$w^* = (a^2 - x^2)(a^2 - y^2).$$

Thus, by the First Extremum Principle,

$$Q \geqslant \frac{k\left[\int_{-a}^{a}\int_{-a}^{a}(a^2 - x^2)(a^2 - y^2)dx\,dy\right]^2}{\int_{-a}^{a}\int_{-a}^{a}4[x^2(a^2 - y^2)^2 + y^2(a^2 - x^2)^2]dx\,dy} = \tfrac{5}{9}ka^4.$$

(Verify the integration if you have the time.)

We conclude that

$$0.555ka^4 \leqslant Q \leqslant 0.667ka^4.$$

The correct value for Q, to three significant figures is $0.562ka^4$. (This result will be obtained from the eigenfunction solution to the problem in *Unit 6, Fourier Series*.)

The construction of the functions w^* and $\mathbf{v}^*$ requires some practice to gain a feel for the most useful choices. It might help to give some idea of the thinking behind the choices in the example. In general, keep your choice of functions as *simple* as possible. For w^* you must select a function which vanishes on the boundary. Thus, for the square pipe we chose

$$w^* = (a^2 - x^2)(a^2 - y^2),$$

which vanishes on $x = \pm a$ and $y = \pm a$. If we think of the analogy with the membrane problem, we can visualize the shape assumed by a membrane attached to a square frame when it is subjected to uniform pressure on one side. We would expect the maximum displacement to occur at the centre; this also occurs for w^*.

For equality to obtain in the upper bound for Q we should have $\mathbf{v}^* = \mathbf{grad}\, w$. Now, in the case of the membrane analogue we would expect, by considerations of symmetry, that $\partial w/\partial x = 0$ on $x = 0$ and $\partial w/\partial y = 0$ on $y = 0$, so that $v_x^* = 0$ on $x = 0$ and $v_y^* = 0$ on $y = 0$. The choice of $\mathbf{v}^* = (v_x^*, v_y^*)$ in the example is the simplest one which has this property in addition to satisfying

$$\text{div}\,\mathbf{v}^* = -k.$$

The following SAQ is lengthy, particularly part (ii). It is incorporated because it enables you to see how accurately the extremum may be determined.

SAQ 16

(i) Show that

$$\mathbf{v}^* = [-\tfrac{1}{2}kx + \mu(x^3 - 3xy^2), -\tfrac{1}{2}ky - \mu(3x^2y - y^3)]$$

is an admissible function in the Second Extremum Principle (i.e. $\operatorname{div}\mathbf{v}^* = -k$) for any value of the constant μ.

(ii) Show that the corresponding upper bound for Q in the square channel of the example is

$$\frac{a^4}{k}\left(\frac{2k^2}{3} + \frac{16k\mu a^2}{15} + \frac{96\mu^2 a^4}{35}\right).$$

(iii) Determine the value of μ which minimizes the upper bound and find the least upper bound.

(Solution on p. 41.)

Here is a similar problem you can try if you have the time.

SAQ 17

Show that $\mathbf{v}^* = (\mu x, (-k - \mu)y)$ is an admissible function for the Second Extremum Principle for any μ. Show that the best μ gives an upper bound for the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

equal to the exact result obtained in SAQ 15.

(Solution on p. 41.)

3.3 THE DIFFUSION EQUATION

3.3.0 Introduction

The partial differential equation which provides a good model for time-dependent heat conduction is also applicable to diffusion, e.g.

(a) diffusion of mixtures of liquids or gases,

(b) diffusion of vorticity in viscous fluids (*vorticity* is the local spin of the fluid).

The one-dimensional form of the equation for heat conduction was derived in *Unit M201 32, The Heat Conduction Equation.* In this section we shall indicate how this can be extended to three dimensions. Before we do that however we need to develop the three-dimensional form of the Divergence Theorem. This in its turn requires some familiarity with non-planar surface integrals and volume integrals which we therefore introduce.

3.3.1 The Divergence Theorem

In Section 3.1.3 we introduced the concept of an integral over a surface contained in the plane. For surfaces in three dimensions we adopt the following approach.

A **surface** in R^3 is defined as the image under a one-to-one map whose domain is a connected subset of R^2 and whose codomain is R^3. It is often convenient to represent the points of the surface by position vectors in R^3. (As for curves, we usually require the surface to be piecewise continuously differentiable, but a precise formulation of this condition for surfaces would take us too far afield.)

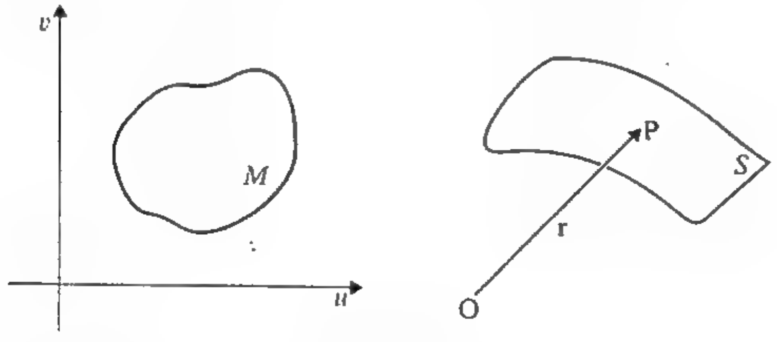
Thus,

$$S = \{\mathbf{r} : \psi(u, v) = \mathbf{r},\ (u, v) \in M\}$$

is a surface for a suitable $M \subset R^2$, and $\mathbf{r} = (x, y, z) \in R^3$. For example,

$$\mathbf{r} = (a \cos u \sin v, a \sin u \sin v, a \cos v) \qquad u \in [0, 2\pi),\ v \in [0, \tfrac{1}{2}\pi]$$

defines a hemisphere of radius a, and is the image of a rectangle in the uv-plane.



If the functions $u \longmapsto x$, $u \longmapsto y$, $u \longmapsto z$, $v \longmapsto x$, $v \longmapsto y$ and $v \longmapsto z$ are continuously differentiable at the point P whose position vector is $\mathbf{r} = (x, y, z)$, then we define the vectors

$$\frac{\partial \mathbf{r}}{\partial u} = \left(\frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u}\right)$$

and

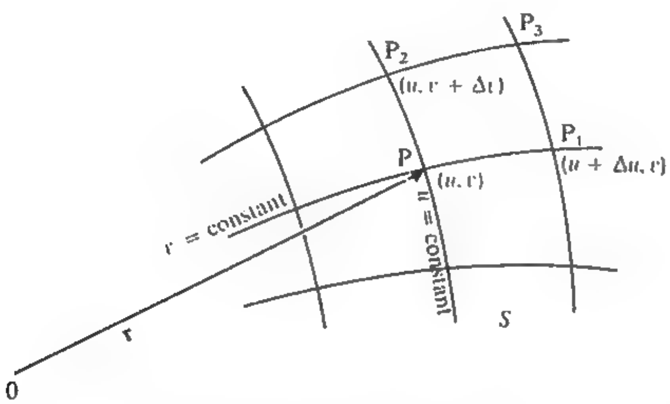
$$\frac{\partial \mathbf{r}}{\partial v} = \left(\frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v}\right).$$

The arrows belonging to these geometric vectors which begin at P are *tangents* to the curves through P along which v and u are constant, respectively. Together they span the *tangent plane* at P (i.e. the plane consisting of arrows at P which are tangent to the surface at P). The vector

$$\mathbf{n} = \left(\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}\right) \Big/ \left|\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}\right|$$

is then a unit vector *normal* to the surface.

We shall now look at a neighbourhood of P in more detail. We shall use the function ψ to define a coordinate system on S: thus we may say that the point P at which $\mathbf{r} = \psi(u, v)$ has coordinates (u, v).



The surface is covered by a mesh of the two families of curves determined by u = constant and v = constant. Let P_1 be the point $\psi(u + \Delta u, v)$ and P_2 the point $\psi(u, v + \Delta v)$. The vectors representing the chords $\overrightarrow{PP_1}$ and $\overrightarrow{PP_2}$ are given, in the first-order Taylor approximation, by

$$\underset{\longrightarrow}{PP_1} = \psi(u + \Delta u, v) - \psi(u, v) \simeq \frac{\partial \mathbf{r}}{\partial u} \Delta u,$$

$$\underset{\longrightarrow}{PP_2} = \psi(u, v + \Delta v) - \psi(u, v) \simeq \frac{\partial \mathbf{r}}{\partial v} \Delta v.$$

Thus, to the same approximation, the area of $PP_1P_3P_2$ is (see Appendix)

$$\Delta S = \left|\underset{\longrightarrow}{PP_1} \times \underset{\longrightarrow}{PP_2}\right| = \left|\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}\right| \Delta u \Delta v.$$

This represents an increment of area of the surface. To integrate a scalar field ϕ over the surface S we define the **surface integral**

$$\iint_S \phi \, dS = \lim_{\Delta S \to 0} \sum \phi \, \Delta S$$

$$= \iint_M \phi \left|\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}\right| du \, dv.$$

Thus any surface integral reduces to an integral over a surface in the plane which can be evaluated as a double integral over Cartesian coordinates. (Where convenient, the double integral in the plane may be transformed to other coordinates, e.g. polar coordinates.) Although we have used S to denote both the surface and an element of area, this should cause no confusion.

The **surface area** of S is defined as $\iint_S dS$; as above this equals

$$\iint_M \left|\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}\right| du \, dv.$$

The **volume integral** $\iiint_D \phi\, dV$ represents the limit of a sum formed by adding together the products $\phi_P \Delta V$ where ϕ_P is the value of ϕ at a point P in the small element ΔV of the domain D in R^3 (see *Unit MST 282 6, Rigid Bodies*, Section 6.2.2). We evaluate such an integral as the *triple integral* over rectangular Cartesian coordinates

$$\iiint \phi\, dx\, dy\, dz,$$

between appropriate limits.

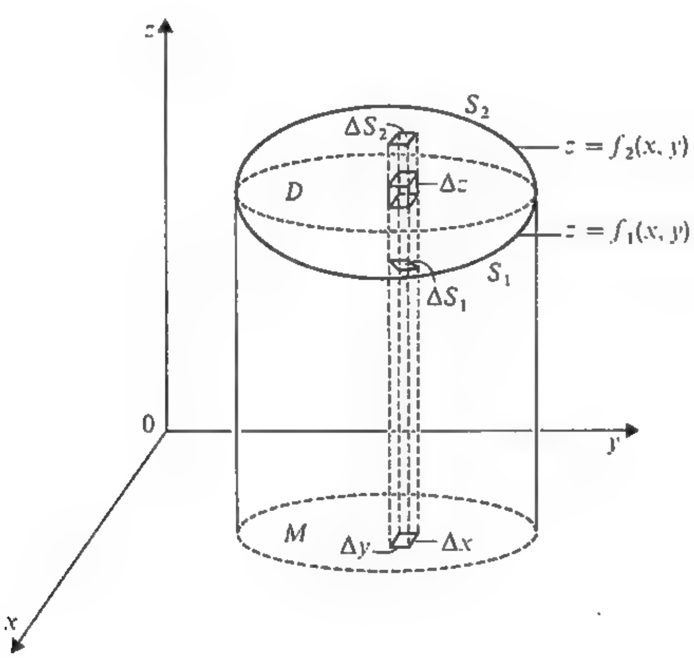
We now have sufficient machinery to state the Divergence Theorem in R^3.

DIVERGENCE THEOREM (IN THREE DIMENSIONS)

Let D be a convex domain in R^3 bounded by the closed surface S. Let the coordinates of the vector field $\mathbf{v}$ have continuous first partial derivatives in D, and let $\mathbf{n}$ be the unit outward normal to S. Then

$$\iiint_D \operatorname{div} \mathbf{v}\, dV = \iint_S \mathbf{v} \cdot \mathbf{n}\, dS.$$

Proof



Divide S into two surfaces S_1 and S_2 as shown and assume that they can be represented by $z = f_1(x, y)$ and $z = f_2(x, y)$. Let $\mathbf{v} = (v_1, v_2, v_3)$ and consider

$$\iiint_D \frac{\partial v_3}{\partial z}\, dV = \iint_M \left[\int_{z=f_1(x,y)}^{z=f_2(x,y)} \frac{\partial v_3}{\partial z}\, dz \right] dx\, dy$$

where M is the projection of D on the xy-plane, i.e. $M = \pi(D)$ where $\pi : (x, y, z) \longmapsto (x, y)$. Thus

$$\iiint_D \frac{\partial v_3}{\partial z}\, dV = \iint_M [v_3]_{z=f_1(x,y)}^{z=f_2(x,y)}\, dx\, dy$$

The surface S_1 is given by

$$\mathbf{r} = (x, y, f_1(x, y)) \qquad (x, y) \in M,$$

and so

$$\frac{\partial \mathbf{r}}{\partial x} = \left(1, 0, \frac{\partial f_1}{\partial x}\right), \quad \frac{\partial \mathbf{r}}{\partial y} = \left(0, 1, \frac{\partial f_1}{\partial y}\right),$$

and

$$\frac{\partial \mathbf{r}}{\partial x} \times \frac{\partial \mathbf{r}}{\partial y} = \left(-\frac{\partial f_1}{\partial x}, -\frac{\partial f_1}{\partial y}, 1\right).$$

The outward unit normal on S_1 is

$$\mathbf{n} = -\left(\frac{\partial \mathbf{r}}{\partial x} \times \frac{\partial \mathbf{r}}{\partial y}\right) \Big/ \left|\frac{\partial \mathbf{r}}{\partial x} \times \frac{\partial \mathbf{r}}{\partial y}\right|$$

The minus sign arises because the z-coordinate of $\mathbf{n}$ must be negative. By definition,

$$\iint_{S_1} v_3\mathbf{k} \cdot \mathbf{n}\, dS = \iint_M v_3\mathbf{k} \cdot \mathbf{n} \left|\frac{\partial \mathbf{r}}{\partial x} \times \frac{\partial \mathbf{r}}{\partial y}\right| dx\, dy$$

$$= -\iint_M v_3\mathbf{k} \cdot \left(\frac{\partial \mathbf{r}}{\partial x} \times \frac{\partial \mathbf{r}}{\partial y}\right) dx\, dy.$$

Now,

$$\mathbf{k} \cdot \frac{\partial \mathbf{r}}{\partial x} \times \frac{\partial \mathbf{r}}{\partial y} = 1,$$

and therefore

$$\iint_{S_1} v_3\mathbf{k} \cdot \mathbf{n}\, dS = -\iint_M v_3|_{z=f_1(x,y)}\, dx\, dy.$$

A parallel argument gives

$$\iint_{S_2} v_3\mathbf{k} \cdot \mathbf{n}\, dS = \iint_M v_3|_{z=f_2(x,y)}\, dx\, dy,$$

and thus

$$\iiint_D \frac{\partial v_3}{\partial z}\, dV = \iint_S v_3\mathbf{k} \cdot \mathbf{n}\, dS.$$

Similarly we may show that

$$\iiint_D \frac{\partial v_1}{\partial x}\, dV = \iint_S v_1\mathbf{i} \cdot \mathbf{n}\, dS,$$

$$\iiint_D \frac{\partial v_2}{\partial y}\, dV = \iint_S v_2\mathbf{j} \cdot \mathbf{n}\, dS.$$

Since

$$\operatorname{div} \mathbf{v} = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z}$$

and

$$\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$$

we obtain

$$\iiint_D \operatorname{div} \mathbf{v}\, dV = \iint_S \mathbf{v} \cdot \mathbf{n}\, dS$$

as required.

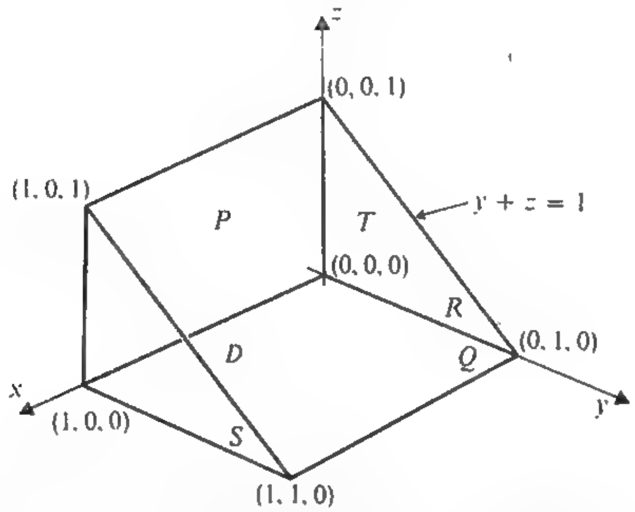
Example

Verify the Divergence Theorem when $\mathbf{v} = (x^3yz, y^2z, -yz^2)$ and D is the domain bounded by the planes $x = 0$, $x = 1$, $z = 0$, $y = 0$, $y + z = 1$.

Solution

We first observe that

$$\operatorname{div} \mathbf{v} = \frac{\partial}{\partial x}(x^3yz) + \frac{\partial}{\partial y}(y^2z) + \frac{\partial}{\partial z}(-yz^2) = 3x^2yz.$$



The domain D is shown in the figure, and the vertices are labelled. To integrate over the whole volume we integrate first with respect to z, between 0 and $1 - y$, then with respect to y between 0 and 1, and finally with respect to x between 0 and 1. Thus

$$\begin{aligned}\iiint_D \operatorname{div} \mathbf{v}\, dV &= 3\int_0^1 \left\{\int_0^1 \left\{\int_0^{1-y} x^2yz\, dz\right\} dy\right\} dx \\ &= 3\int_0^1\int_0^1 \tfrac{1}{2}x^2y(1-y)^2 dy\, dx \\ &= \frac{3}{2}\int_0^1 \tfrac{1}{12}x^2 dx = \tfrac{1}{24}.\end{aligned}$$

We now have to evaluate $\iint \mathbf{v}\cdot\mathbf{n}\, dS$ over the surface of D. First note that the unit outward normals $\mathbf{n}$ on the respective faces (which are denoted by P, Q, R, S, T) are as follows.

$$\begin{aligned} y = 0 &: \mathbf{n}_P = -\mathbf{j} \\ z = 0 &: \mathbf{n}_Q = -\mathbf{k} \\ x = 0 &: \mathbf{n}_R = -\mathbf{i} \\ x = 1 &: \mathbf{n}_S = \mathbf{i} \\ y + z = 1 &: \mathbf{n}_T = (\mathbf{j} + \mathbf{k})/\sqrt{2} \end{aligned}$$

We thus obtain

$$\iint_P \mathbf{v}\cdot\mathbf{n}_P\, dS = \iint_Q \mathbf{v}\cdot\mathbf{n}_Q\, dS = \iint_R \mathbf{v}\cdot\mathbf{n}_R\, dS = 0.$$

$$\iint_S \mathbf{v}\cdot\mathbf{n}_S\, dS = \int_0^1\left\{\int_0^{1-y} yz\, dz\right\}dy = \frac{1}{2}\int_0^1 y(1-y)^2 dy = \tfrac{1}{24}.$$

$$\begin{aligned}\iint_T \mathbf{v}\cdot\mathbf{n}_T\, dS &= \frac{1}{\sqrt{2}}\int_0^1\int_0^1 [y^2z - yz^2]_{z=1-y}\sqrt{2}\, dx\, dy \\ &= \int_0^1\int_0^1 [y^2(1-y) - y(1-y)^2]\, dx\, dy = 0.\end{aligned}$$

Thus, we see that

$$\iiint_D \operatorname{div} \mathbf{v}\, dV = \tfrac{1}{24} = \iint_{\text{surface of } D} \mathbf{v} \cdot \mathbf{n}\, dS.$$

SAQ 18

Verify the Divergence Theorem when $\mathbf{v} = (xy, yz, zx)$ and D is the cube bounded by $x = 0, x = 1, y = 0, y = 1, z = 0, z = 1$.

(Solution on p. 42.)

3.3.2 Derivation of the Heat Equation

Suppose the material in which heat is flowing occupies the domain $D \subset R^3$ and let $u(x, y, z, t)$ be the temperature at the point $(x, y, z) \in D$ at time $t \geqslant 0$. Fourier's heat transfer law is

$$\mathbf{J} = -K \operatorname{\mathbf{grad}} u, \tag{1}$$

where $\mathbf{J}$ is called the *heat current* vector and represents the rate and direction at which heat is flowing at a point, and K is the *thermal conductivity* of the material which is assumed (for our purposes) to be constant in time and space. Fourier's Law states that heat flows in the direction of greatest change in temperature from points of higher temperature to points of lower temperature (this accounts for the minus sign).

Consider now any subdomain D_0 of D. The *heat content* $H(t)$ of D_0 is defined to be

$$H(t) = c\rho \iiint_{D_0} u(x, y, z, t)\, dx\, dy\, dz, \tag{2}$$

where ρ is the density and c the *specific heat* of the material; both are assumed to be constant in space and time. (The specific heat is the heat required to raise the temperature of unit mass of the material by one unit.) The heat equation follows from the Principle of Conservation of Heat (a form of the Law of Conservation of Energy) which states that the rate of heat accumulating in D_0 equals the rate of heat entering through the boundary; in mathematical terms

$$\frac{dH}{dt} = -\iint_{C_0} \mathbf{J} \cdot \mathbf{n}\, dS,$$

where the surface C_0 is the boundary of D_0 and $\mathbf{n}$ is the unit outward normal to C_0. Thus, from Equations (1) and (2), and differentiating under the integral sign,

$$\begin{aligned} c\rho \iiint_{D_0} \frac{\partial u}{\partial t}\, dx\, dy\, dz &= K \iint_{C_0} (\operatorname{\mathbf{grad}} u) \cdot \mathbf{n}\, dS \\ &= K \iiint_{D_0} \nabla^2 u\, dx\, dy\, dz \end{aligned}$$

by the Divergence Theorem. Finally

$$\iiint_{D_0} \left\{ c\rho \frac{\partial u}{\partial t} - K\nabla^2 u \right\} dx\, dy\, dz = 0;$$

since this is true for *any* domain D_0 we infer that the integrand vanishes, giving the **heat** or **diffusion equation**

$$\frac{\partial u}{\partial t} = k\nabla^2 u,$$

where $k = K/c\rho$.

If the temperature is steady, then u is independent of t and must satisfy Laplace's equation

$$\nabla^2 u = 0.$$

There are three main types of spatial boundary condition which may be satisfied by the temperature:

(i) u is a prescribed function of t at each point on C_0;

(ii) $\mathbf{n} \cdot \mathbf{grad}\, u$ is prescribed on C_0 (the boundary is said to be **insulated** at those points at which $\mathbf{n} \cdot \mathbf{grad}\, u = 0$);

(iii) $\mathbf{n} \cdot \mathbf{grad}\, u + (u - u_0)h = 0$ where h and u_0 are constants (this represents radiation from the surface into a medium of fixed temperature).

3.3.3 Uniqueness and Maximum Principle

READ W: page 59, line 1 to page 61, line 5

Notes

(i) *W: page 59, lines −12 to −7*
This represents a statement of the uniqueness theorem for $u = u_1 - u_2$ where u_1 and u_2 are any two solutions to the heat equation in D such that $u_1 = u_2$ on C for $t \in [0, \bar{t}]$ and in D for $t = 0$. It is then shown that $u \equiv 0$ in D for $0 \leqslant t \leqslant \bar{t}$.

(ii) *W: page 59, line −4*
Green's Theorem in three dimensions is a special case of the three-dimensional Divergence Theorem (Section 3.3.1) with $\mathbf{v} = u\, \mathbf{grad}\, u$.

SAQ 19

Show formally that, for any function $f: R \longrightarrow R$ which is continuous and whose values are bounded,

$$u(x, t) = \frac{1}{2(kt\pi)^{\frac{1}{2}}} \int_{-\infty}^{\infty} e^{-((\xi - x)^2/4kt)} f(\xi)\, d\xi$$

satisfies the one-dimensional heat equation

$$\frac{\partial u}{\partial t} - k\frac{\partial^2 u}{\partial x^2} = 0.$$

(You should assume that you may differentiate under the integral sign.)

By substituting $\xi - x = 2\eta(tk)^{\frac{1}{2}}$ in the integral show that

$$\lim_{t \to 0} u(x, t) = f(x),$$

under the assumption that, for the integrand concerned,

$$\lim_{t \to 0} \int_{-\infty}^{\infty} = \int_{-\infty}^{\infty} \lim_{t \to 0}$$

HINT: $\displaystyle\int_{-\infty}^{\infty} e^{-\eta^2}\, d\eta = \sqrt{\pi}.$

Describe a heat conduction problem to which the function given above is a solution.

(Solution on p. 43.)

SAQ 20

W: page 61, Exercise 2.

(Solution on p. 44.)

3.4 SUMMARY

We have covered the following main points in this text and the associated reading passages:

(a) uniqueness of solutions of *Poisson's equation* and the *heat equation* with appropriate boundary conditions;

(b) the maximum principle for the same two equations;

(c) the notion of a *properly posed* problem.

The application of extremum principles to flow in a pipe was introduced and we showed how bounds could be obtained for the flux through the pipe.

The following results were introduced:

Stokes' Theorem in the plane,
the Divergence Theorem in two and three dimensions, and
Green's Theorem in two and three dimensions.

In the course of discussing these theorems we defined *curves* and *surfaces* in R^2 and R^3 and developed techniques for evaluating *line integrals*, *surface integrals* and *volume integrals*.

3.5 SOLUTIONS TO SELF-ASSESSMENT QUESTIONS

Solution to SAQ 1

(i) $\sqrt{3}$

(ii) $3\sqrt{3}$

(iii) 0

(iv) $3\mathbf{j} - 3\mathbf{k}$

(v) 6

(vi) $6\mathbf{i} - 3\mathbf{j} - 3\mathbf{k}$

Solution to SAQ 2

(a) $3(x^2 + y^2 + z^2)^{\frac{1}{2}}(x\mathbf{i} + y\mathbf{j} + z\mathbf{k})$

(b) $yz\mathbf{i} + zx\mathbf{j} + xy\mathbf{k}$

(c) $2x\mathbf{i} + 2yz\mathbf{j} + (y^2 + 3z^2)\mathbf{k}$

Solution to SAQ 3

$\operatorname{div} \mathbf{F} = 2y + 6xz.$

Solution to SAQ 4

A unit vector in the direction given is

$$\mathbf{e} = \frac{(\mathbf{i} - \mathbf{j} - \mathbf{k})}{(1 + 1 + 1)^{\frac{1}{2}}} = \frac{1}{\sqrt{3}}(\mathbf{i} - \mathbf{j} - \mathbf{k}).$$

At (1, 2, 1)

$$\mathbf{grad}\,\phi = 2\mathbf{i} + \mathbf{j} + 2\mathbf{k}.$$

Therefore, at (1, 2, 1)

$$\frac{\partial \phi}{\partial e} = \mathbf{e} \cdot \mathbf{grad}\,\phi = \frac{1}{\sqrt{3}}(2 - 1 - 2) = -\frac{1}{\sqrt{3}}.$$

Solution to SAQ 5

By the chain rule for functions of several variables, we have

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r}$$

and

$$\frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \theta}.$$

Since

$$x = r\cos\theta, \quad y = r\sin\theta,$$

we have

$$\frac{\partial x}{\partial r} = \cos\theta = \frac{x}{r}, \quad \frac{\partial y}{\partial r} = \sin\theta = \frac{y}{r},$$

and

$$\frac{\partial x}{\partial \theta} = -r\sin\theta = -y, \quad \frac{\partial y}{\partial \theta} = r\cos\theta = x.$$

Thus,

$$r\frac{\partial u}{\partial r} = x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}$$

and

$$\frac{\partial u}{\partial \theta} = -y\frac{\partial u}{\partial x} + x\frac{\partial u}{\partial y}.$$

Hence,

$$r\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) = \left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right),$$

i.e.

$$r^2\frac{\partial^2 u}{\partial r^2} + r\frac{\partial u}{\partial r} = x^2\frac{\partial^2 u}{\partial x^2} + x\frac{\partial u}{\partial x} + yx\frac{\partial^2 u}{\partial y \partial x} + xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} + y\frac{\partial u}{\partial y}.$$

and

$$\frac{\partial^2 u}{\partial \theta^2} = \left(-y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}\right)\left(-y\frac{\partial u}{\partial x} + x\frac{\partial u}{\partial y}\right)$$

$$= y^2\frac{\partial^2 u}{\partial x^2} - x\frac{\partial u}{\partial x} - xy\frac{\partial^2 u}{\partial y \partial x} - yx\frac{\partial^2 u}{\partial x \partial y} - y\frac{\partial u}{\partial y} + x^2\frac{\partial^2 u}{\partial y^2}.$$

Therefore,

$$r^2\frac{\partial^2 u}{\partial r^2} + r\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial \theta^2} = (x^2 + y^2)\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

and the result follows, since $r^2 = x^2 + y^2$.

Let $w: r \longmapsto u$; then, with $G(r, \theta) = r$ and assuming that u has radial symmetry the equation becomes the ordinary differential equation

$$w''(r) + \frac{1}{r}w'(r) = -r$$

or

$$\frac{d}{dr}(rw'(r)) = -r^2.$$

Integration gives

$$rw'(r) = -\tfrac{1}{3}r^3 + A,$$

where A is an arbitrary constant. A second integration gives

$$w(r) = -\tfrac{1}{9}r^3 + A \ln r + B,$$

where B is a further arbitrary constant. Since $\ln r$ is unbounded as $r \rightsquigarrow 0^+$ (i.e. as r tends to zero through positive values), we put $A = 0$, whilst B is given by

$$0 = -\tfrac{1}{9}a^3 + B.$$

Thus the solution is

$$u = \tfrac{1}{9}(a^3 - r^3).$$

Solution to SAQ 6

The derivatives of Ψ at (r, θ) are given by

$$\frac{\partial \Psi}{\partial r}(r, \theta) = b\left(1 + \frac{a^2}{r^2}\right)\sin\theta,$$

$$\frac{\partial^2 \Psi}{\partial r^2}(r, \theta) = \frac{-2ba^2}{r^3}\sin\theta,$$

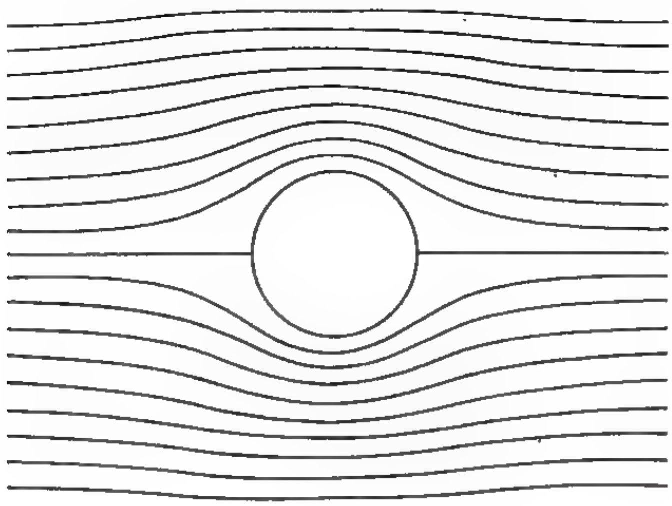
and

$$\frac{\partial^2 \Psi}{\partial \theta^2}(r, \theta) = -b\left(r - \frac{a^2}{r}\right)\sin\theta;$$

thus

$$\frac{\partial^2\Psi}{\partial r^2}+\frac{1}{r}\frac{\partial\Psi}{\partial r}+\frac{1}{r^2}\frac{\partial^2\Psi}{\partial\theta^2}=0.$$

The figure shows some typical curves along which the value of Ψ is constant near the circle $r = a$.



For large r,

$$\Psi(r,\theta) \simeq br\sin\theta = by.$$

Thus the curves approach the straight lines $y =$ constant.

In the context of fluid dynamics, the velocity field for large r is approximately $(b, 0)$, a uniform stream. The curves in the figure represent streamlines; note that $\Psi(r,0) = \Psi(r,\pi) = \Psi(a,\theta) = 0$ for $r > a$ and $0 \leqslant \theta < 2\pi$.

Our choice of Ψ represents uniform flow past a circular cylinder.

Solution to SAQ 7

$$e^{-\sqrt{n}}\cosh(4n+1)\pi y = \tfrac{1}{2}e^{-\sqrt{n}}\left(e^{(4n+1)\pi y} - e^{-(4n+1)\pi y}\right);$$

For large n and $y \neq 0$ this behaves like

$$e^{4n\pi|y|-\sqrt{n}}.$$

For

$$n > \frac{1}{16\pi^2 y^2},$$

this has a positive exponent which increases without limit as n becomes large. Hence the expression is unbounded for large n.

Solution to SAQ 8

$$\operatorname{div}\mathbf{I} = \operatorname{div}(\sigma\mathbf{E}) = -\operatorname{div}(\sigma\,\mathbf{grad}\,u).$$

Thus u satisfies

$$\operatorname{div}(\sigma\,\mathbf{grad}\,u) = 0,$$

i.e.

$$\frac{\partial}{\partial x}\left(\sigma\frac{\partial u}{\partial x}\right)+\frac{\partial}{\partial y}\left(\sigma\frac{\partial u}{\partial y}\right)+\frac{\partial}{\partial z}\left(\sigma\frac{\partial u}{\partial z}\right)=0$$

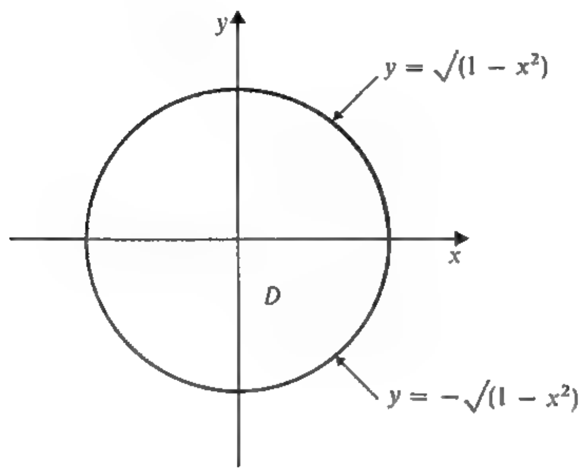
or

$$\sigma\nabla^2 u+\frac{\partial\sigma}{\partial x}\frac{\partial u}{\partial x}+\frac{\partial\sigma}{\partial y}\frac{\partial u}{\partial y}+\frac{\partial\sigma}{\partial z}\frac{\partial u}{\partial z}=0.$$

The second derived functions occur in the term $\sigma\nabla^2 u$ which is the product of $\sigma(\neq 0)$ and the Laplace operator acting on u. Thus the equation must be elliptic.

Solution to SAQ 9

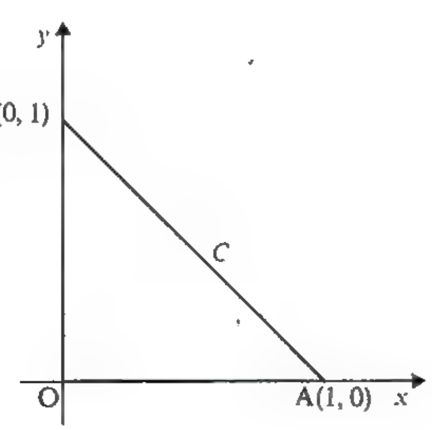
$$\begin{aligned}\operatorname{div}\mathbf{v} &= \frac{\partial}{\partial x}(x\sqrt{1-x^2}) + \frac{\partial}{\partial y}(y\sqrt{1-x^2})\\ &= \frac{2-3x^2}{\sqrt{(1-x^2)}}.\end{aligned}$$



$$\begin{aligned}\iint_D \operatorname{div}\mathbf{v}\,dA &= \int_{-1}^{1}\int_{-\sqrt{(1-x^2)}}^{\sqrt{(1-x^2)}} \frac{2-3x^2}{\sqrt{(1-x^2)}}\,dy\,dx\\ &= \int_{-1}^{1} \frac{(2-3x^2)}{\sqrt{(1-x^2)}}\Big[\,y\,\Big]_{-\sqrt{(1-x^2)}}^{\sqrt{(1-x^2)}}\,dx\\ &= \int_{-1}^{1} 2(2-3x^2)dx = 4.\end{aligned}$$

(Integrating with respect to x first would have been rather tedious!)

Solution to SAQ 10



$$\int_{OA} \mathbf{v}\cdot\mathbf{n}\,ds = \int_0^1 [2y^2]_{y=0}dx = 0,$$

since $\mathbf{n} = -\mathbf{j}$ and $y = 0$ on OA.

$$\int_{BO} \mathbf{v}\cdot\mathbf{n}\,ds = \int_1^0 [5x]_{x=0}dy = 0,$$

since $\mathbf{n} = -\mathbf{i}$ and $x = 0$ on BO.

The line AB has the equation $x + y = 1$. In this case the line is given parametrically by

$$x = 1 - y \qquad y \in [0, 1].$$

The outward normal is $\mathbf{n} = (\mathbf{i} + \mathbf{j})/\sqrt{2}$ and

$$\begin{aligned}\frac{ds}{dy} &= \left\{\left(\frac{dx}{dy}\right)^2 + 1\right\}^{\frac{1}{2}}\\ &= \sqrt{2}.\end{aligned}$$

Therefore

$$\begin{aligned}\int_{AB} \mathbf{v}\cdot\mathbf{n}\,ds &= \int_0^1 [-5(1-y) - 2y^2]dy\\ &= -5 + \tfrac{5}{2} - \tfrac{2}{3} = -\tfrac{19}{6}.\end{aligned}$$

$$\begin{aligned}\iint_D \operatorname{div}\mathbf{v}\,dA &= \int_0^1\int_0^{1-x}\left[\frac{\partial}{\partial x}(-5x) + \frac{\partial}{\partial y}(-2y^2)\right]dy\,dx\\ &= \int_0^1\int_0^{1-x}[-5-4y]dy\,dx\\ &= \int_0^1[-5(1-x) - 2(1-x)^2]dx\\ &= \int_0^1(-7 + 9x - 2x^2)dx\\ &= -\tfrac{19}{6}.\end{aligned}$$

Solution to SAQ 11

(a) $\operatorname{div}(a\mathbf{b}) = \dfrac{\partial(ab_x)}{\partial x} + \dfrac{\partial(ab_y)}{\partial y}$, where $\mathbf{b} = b_x\mathbf{i} + b_y\mathbf{j}$

$$\begin{aligned}&= \frac{\partial a}{\partial x}b_x + a\frac{\partial b_x}{\partial x} + \frac{\partial a}{\partial y}b_y + a\frac{\partial b_y}{\partial y}\\ &= a\operatorname{div}\mathbf{b} + \mathbf{grad}\,a\cdot\mathbf{b}.\end{aligned}$$

(b) The Divergence Theorem gives

$$\begin{aligned}\iint_D \operatorname{div}(u\,\mathbf{grad}\,v)\,dA &= \oint_C u\,\mathbf{grad}\,v\cdot\mathbf{n}\,ds\\ &= \oint_C u\frac{\partial v}{\partial n}ds.\end{aligned}$$

Using the identity $\operatorname{div}(a\mathbf{b}) = a\operatorname{div}\mathbf{b} + \mathbf{grad}\,a\cdot\mathbf{b}$ in the left-hand side:

$$\iint_D \mathbf{grad}\,u\cdot\mathbf{grad}\,v\,dA + \iint_D u\nabla^2 v\,dA = \oint_C u\frac{\partial v}{\partial n}ds.$$

Note that if $u = v$ we obtain Green's Theorem (*W: page 53*) as a special case.

(c) Interchanging u and v in part (b) we obtain

$$\iint_D \mathbf{grad}\,v\cdot\mathbf{grad}\,u\,dA + \iint_D v\nabla^2 u\,dA = \oint_C v\frac{\partial u}{\partial n}ds.$$

Subtracting this from the result of part (b) we obtain the required result.

Solution to SAQ 12

Let u_1 and u_2 be two solutions, which satisfy Laplace's equation and the given boundary condition. If $v = u_1 - u_2$ then v satisfies the homogeneous problem

$$\nabla^2 v = 0 \quad \text{in } D,$$

$$\frac{\partial v}{\partial n} + hv = 0 \quad \text{on } C.$$

We apply the identity (11.2) on *W: page 53* for v, so that

$$-\oint_C v\frac{\partial v}{\partial n}\,ds + \iint_D |\mathbf{grad}\, v|^2 dx\, dA = 0.$$

Substituting for $\partial v/\partial n$ on C gives

$$\oint_C hv^2 ds + \iint_D |\mathbf{grad}\, v|^2\, dA = 0.$$

Since $h > 0$ on C, the integrands are strictly positive unless v takes the value zero throughout C and is constant in D. By continuity, v is the zero function on $D \cup C$ and so $u_1 = u_2$.

Solution to SAQ 13

Let $u = u_2 - u_1$. It follows that $\nabla^2 u = 0$ and $u > 0$ on C. The minimum value of u must occur on the boundary and further

$$\min_C u > 0.$$

Thus

$$u \geqslant \min_C u > 0$$

in D. Thus $u_1 < u_2$ in D.

Solution to SAQ 14

If

$$w = \tfrac{1}{4}k(a^2 - x^2 - y^2),$$

then

$$\frac{\partial^2 w}{\partial x^2} = -\tfrac{1}{2}k, \qquad \frac{\partial^2 w}{\partial y^2} = -\tfrac{1}{2}k.$$

Thus $\nabla^2 w = -k$, so that w satisfies Equation (1). It is easily shown that Equation (2) is satisfied, since $x^2 + y^2 = a^2$ on C. The expression for w gives *the* velocity since the solution is unique.

The flux is given by

$$Q = \iint_D w\, dA = \int_0^{2\pi} \left\{ \int_0^a wr\, dr \right\} d\theta$$

in polar coordinates, where $w = \frac{1}{4}k(a^2 - r^2)$. (For our problem, integration is easier in polar coordinates than in Cartesian coordinates, since the domain is a circle.) Thus the flux is

$$Q = \tfrac{1}{4}k \int_0^{2\pi} \left\{ \int_0^a (ra^2 - r^3)dr \right\} d\theta$$

$$= \tfrac{1}{2}k\pi \left[\tfrac{1}{2}r^2a^2 - \tfrac{1}{4}r^4 \right]_0^a$$

$$= \tfrac{1}{8}k\pi a^4.$$

Solution to SAQ 15

The solution for the circular pipe (SAQ 14) suggests (if you are good at guessing) a solution of the form

$$w = c\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right),$$

which clearly satisfies the boundary condition (Equation (2)). It also satisfies $\nabla^2 w = -k$ if

$$-2c\left(\frac{1}{a^2} + \frac{1}{b^2}\right) = -k,$$

or

$$c = \frac{k}{2}\cdot\frac{a^2b^2}{a^2+b^2}.$$

So the speed of flow is given by

$$w = \frac{k}{2}\cdot\frac{a^2b^2}{a^2+b^2}\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right).$$

The flux is given by

$$\begin{aligned} Q &= c\int_{-a}^{a}\left\{\int_{-b\sqrt{(1-x^2/a^2)}}^{b\sqrt{(1-x^2/a^2)}}\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right)dy\right\}dx \\ &= c\int_{-a}^{a}\left[\left(1 - \frac{x^2}{a^2}\right)y - \frac{y^3}{3b^2}\right]_{y=-b\sqrt{(1-x^2/a^2)}}^{y=b\sqrt{(1-x^2/a^2)}}dx \\ &= 2bc\int_{-a}^{a}\frac{2}{3}\left(1 - \frac{x^2}{a^2}\right)^{\frac{3}{2}}dx. \end{aligned}$$

Now, if we make the substitution

$$x = a\sin u,$$

so that

$$\frac{dx}{du} = a\cos u,$$

then

$$\begin{aligned} \int_{-a}^{a}\left(1 - \frac{x^2}{a^2}\right)^{\frac{3}{2}}dx &= a\int_{-\pi/2}^{\pi/2}\cos^3 u\,.\,\cos u\,du \\ &= \frac{a}{8}\int_{-\pi/2}^{\pi/2}(3 + 4\cos 2u + \cos 4u)du \qquad \text{using the hint} \\ &= \frac{a}{8}[3u + 2\sin 2u + \tfrac{1}{4}\sin 4u]_{-\pi/2}^{\pi/2} \\ &= 3\pi a/8. \end{aligned}$$

Therefore

$$\begin{aligned} Q &= \frac{4bc}{3}\cdot\frac{3\pi a}{8} \\ &= \frac{k}{2}\cdot\frac{a^2b^2}{a^2+b^2}\cdot\frac{\pi ab}{2} \\ &= \frac{\pi k a^3 b^3}{4(a^2+b^2)}, \end{aligned}$$

as required.

Solution to SAQ 16

(i) $\operatorname{div} \mathbf{v}^* = -\tfrac{1}{2}k + 3\mu x^2 - 3\mu y^2 - \tfrac{1}{2}k - 3\mu x^2 + 3\mu y^2$
$= -k.$

and so $\mathbf{v}^*$ is admissible.

(ii) The upper bound on Q for the square channel is

$$\frac{1}{k}\iint_D v^{*2}\,dA = \frac{1}{k}\int_{-a}^{a}\int_{-a}^{a}\{[-\tfrac{1}{2}kx+\mu(x^3-3xy^2)]^2+[-\tfrac{1}{2}ky-\mu(3x^2y-y^3)]^2\}\,dx\,dy$$

$$=\frac{1}{k}\int_{-a}^{a}\int_{-a}^{a}[\tfrac{1}{4}k^2x^2-k\mu(x^4-3x^2y^2)+\mu^2(x^6-6x^4y^2+9x^2y^4)$$
$$+\tfrac{1}{4}k^2y^2+k\mu(3x^2y^2-y^4)+\mu^2(9x^4y^2-6x^2y^4+y^6)]dx\,dy$$

$$=\frac{1}{k}\int_{-a}^{a}\int_{-a}^{a}[\tfrac{1}{4}k^2(x^2+y^2)-k\mu(x^4-6x^2y^2+y^4)+\mu^2(x^6+3x^4y^2+3x^2y^4+y^6)]dx\,dy$$

$$=\frac{1}{k}\int_{-a}^{a}\left[\tfrac{1}{2}k^2\left(\frac{a^3}{3}+ay^2\right)-2k\mu\left(\frac{a^5}{5}-2a^3y^2+ay^4\right)+2\mu^2\left(\frac{a^7}{7}+\frac{3a^5y^2}{5}+a^3y^4+ay^6\right)\right]dy$$

$$=\frac{1}{k}\left[k^2\left(\frac{a^4}{3}+\frac{a^4}{3}\right)-4k\mu\left(\frac{a^6}{5}-\frac{2a^6}{3}+\frac{a^6}{5}\right)+4\mu^2\left(\frac{a^8}{7}+\frac{a^8}{5}+\frac{a^8}{5}+\frac{a^8}{7}\right)\right]$$

$$=\frac{a^4}{k}(\tfrac{2}{3}k^2+\tfrac{16}{15}k\mu a^2+\tfrac{96}{35}\mu^2a^4) = p(\mu) \quad \text{say.}$$

(iii) The best μ occurs where $p'(\mu) = 0$ so that $p(\mu)$ is a minimum, i.e. where

$$\mu a^2 = -7k/36.$$

Thus the least upper bound is

$$ka^4[\tfrac{2}{3} - \tfrac{16}{15}\cdot\tfrac{7}{36} + \tfrac{96}{35}\cdot(\tfrac{7}{36})^2] = \tfrac{76}{135}ka^4 = 0.563ka^4.$$

Compare this upper bound with the correct value given in the text.

The choice of admissible function in the text corresponds to $\mu = 0$.

Solution to SAQ 17

$\mathbf{v}^*$ is admissible since

$$\operatorname{div} \mathbf{v}^* = \mu - k - \mu = -k.$$

The upper bound is given by

$$\frac{1}{k}\iint_D v^{*2}\,dx\,dy = \frac{1}{k}\int_{-a}^{a}\left\{\int_{-b\sqrt{(1-x^2/a^2)}}^{b\sqrt{(1-x^2/a^2)}}[\mu^2x^2+(k+\mu)^2y^2]dy\right\}dx$$

$$=\frac{1}{k}\int_{-a}^{a}\left[\mu^2x^2y+\tfrac{1}{3}(k+\mu)^2y^3\right]_{-b\sqrt{(1-x^2/a^2)}}^{b\sqrt{(1-x^2/a^2)}}dx$$

$$=\frac{2}{k}\int_{-a}^{a}\left[\mu^2x^2b\left(1-\frac{x^2}{a^2}\right)^{\frac{1}{2}}+\tfrac{1}{3}(k+\mu)^2b^3\left(1-\frac{x^2}{a^2}\right)^{\frac{3}{2}}\right]dx.$$

We use the substitution $x = a\sin u$ to obtain

$$\frac{1}{k}\iint_D v^{*2}\,dx\,dy = \frac{2ab}{k}\int_{-\pi/2}^{\pi/2}[\mu^2a^2\sin^2u\cos u+\tfrac{1}{3}(k+\mu)^2b^2\cos^3u]\cos u\,du$$

$$=\frac{2ab}{k}\int_{-\pi/2}^{\pi/2}[\tfrac{1}{4}\mu^2a^2\sin^2 2u+\tfrac{1}{12}(k+\mu)^2b^2(1+\cos 2u)^2]du$$

$$=\frac{2ab}{k}\int_{-\pi/2}^{\pi/2}[\tfrac{1}{8}\mu^2a^2(1-\cos 4u)+\tfrac{1}{24}(k+\mu)^2b^2(3+4\cos 2u+\cos 4u)]du$$

$$=\frac{2ab\pi}{k}[\tfrac{1}{8}\mu^2a^2+\tfrac{1}{8}(k+\mu)^2b^2]=p(\mu), \quad \text{say}$$

The least upper bound occurs where $p'(\mu) = 0$, that is, where

$$\mu a^2 + (k + \mu)b^2 = 0.$$

Thus

$$\mu = -kb^2/(a^2 + b^2),$$

and the least upper bound is

$$\frac{2ab\pi k}{8}\left[\frac{a^2b^4}{(a^2 + b^2)^2} + \frac{b^2a^4}{(a^2 + b^2)^2}\right] = \frac{\pi a^3b^3k}{4(a^2 + b^2)},$$

which is equal to the true value for Q (see SAQ 15).

Note that if we are interested only in the best value of μ we can proceed directly using the value of w obtained in SAQ 15. For equality in the upper bound we require

$$v^* = \mathbf{grad}\, w = \left(-\frac{kb^2}{a^2 + b^2}x, -\frac{ka^2}{a^2 + b^2}y\right)$$

$$= (\mu x, (-k - \mu)y)$$

when

$$\mu = -kb^2/(a^2 + b^2).$$

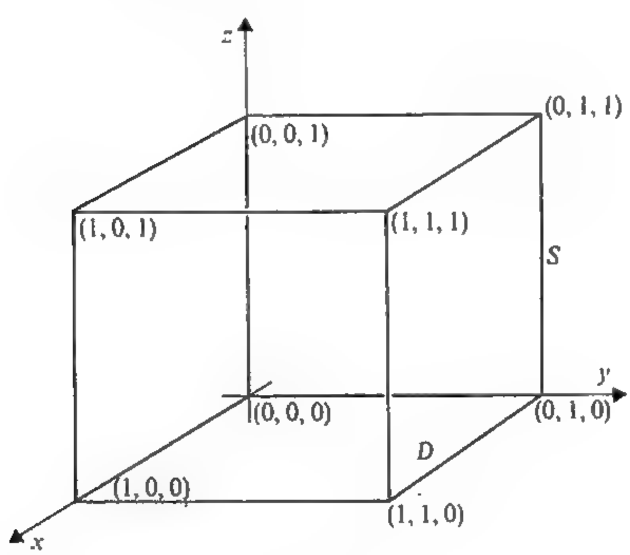
Solution to SAQ 18

$$\iiint_D \operatorname{div} \mathbf{v}\, dV = \int_0^1\int_0^1\int_0^1 (x + y + z)dx\, dy\, dz$$

$$= \int_0^1\int_0^1 (\tfrac{1}{2} + y + z)dy\, dz$$

$$= \int_0^1 (\tfrac{1}{2} + \tfrac{1}{2} + z)dz$$

$$= \tfrac{1}{2} + \tfrac{1}{2} + \tfrac{1}{2} = \tfrac{3}{2}.$$



We now evaluate $\iint_S \mathbf{v} \cdot \mathbf{n}\, dS$ over the six faces.

For $x = 0$: $\mathbf{n} = -\mathbf{i}$, $\mathbf{v} = (0, yz, 0)$, $\iint \mathbf{v} \cdot \mathbf{n}\, dS = 0$.

For $y = 0$: $\mathbf{n} = -\mathbf{j}$, $\mathbf{v} = (0, 0, zx)$, $\iint \mathbf{v} \cdot \mathbf{n}\, dS = 0$.

For $z = 0$: $\mathbf{n} = -\mathbf{k}$, $\mathbf{v} = (xy, 0, 0)$, $\iint \mathbf{v} \cdot \mathbf{n}\, dS = 0$.

For $x = 1 : \mathbf{n} = \mathbf{i}$, $\mathbf{v} = (y, yz, z)$, $\iint \mathbf{v} \cdot \mathbf{n}\, dS = \int_0^1 \int_0^1 y\, dy\, dz = \frac{1}{2}$.

For $y = 1 : \mathbf{n} = \mathbf{j}$, $\mathbf{v} = (x, z, zx)$, $\iint \mathbf{v} \cdot \mathbf{n}\, dS = \int_0^1 \int_0^1 z\, dz\, dx = \frac{1}{2}$.

For $z = 1 : \mathbf{n} = \mathbf{k}$, $\mathbf{v} = (xy, y, x)$, $\iint \mathbf{v} \cdot \mathbf{n}\, dS = \int_0^1 \int_0^1 x\, dy\, dx = \frac{1}{2}$.

Thus

$$\iint_S \mathbf{v} \cdot \mathbf{n}\, dS = \tfrac{1}{2} + \tfrac{1}{2} + \tfrac{1}{2} = \tfrac{3}{2} = \iiint_D \operatorname{div} \mathbf{v}\, dV.$$

Solution to SAQ 19

Assuming that we may differentiate under the integral sign, we obtain

$$\frac{\partial u}{\partial t} = \frac{1}{2(k\pi)^{\frac{1}{2}}} \int_{-\infty}^{\infty} \left[-\frac{1}{2t^{\frac{3}{2}}} + \frac{(\xi - x)^2}{4kt^{\frac{5}{2}}} \right] e^{-\{(\xi - x)^2/4kt\}} f(\xi)\, d\xi,$$

$$\frac{\partial u}{\partial x} = \frac{1}{2(k\pi)^{\frac{1}{2}}} \int_{-\infty}^{\infty} \frac{(\xi - x)}{2kt^{\frac{3}{2}}} e^{-\{(\xi - x)^2/4kt\}} f(\xi) d\xi,$$

and

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{2(k\pi)^{\frac{1}{2}}} \int_{-\infty}^{\infty} \left[\frac{(\xi - x)^2}{4k^2 t^{\frac{5}{2}}} - \frac{1}{2kt^{\frac{3}{2}}} \right] e^{-\{(\xi - x)^2/4kt\}} f(\xi) d\xi;$$

so that

$$\frac{\partial u}{\partial t} - k \frac{\partial^2 u}{\partial x^2} = 0.$$

The substitution

$$\xi - x = 2\eta(tk)^{\frac{1}{2}}$$

gives

$$u(x, t) = \frac{1}{2(kt\pi)^{\frac{1}{2}}} \int_{-\infty}^{\infty} e^{-\eta^2} f(x + 2\eta(tk)^{\frac{1}{2}}) \,.\, 2(tk)^{\frac{1}{2}} d\eta$$

$$= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-\eta^2} f(x + 2\eta(tk)^{\frac{1}{2}}) d\eta.$$

Thus under the assumption that the limit and the infinite integration may be interchanged,

$$\lim_{t \to 0} u(x, t) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-\eta^2} \lim_{t \to 0} f(x + 2\eta(tk)^{\frac{1}{2}}) d\eta$$

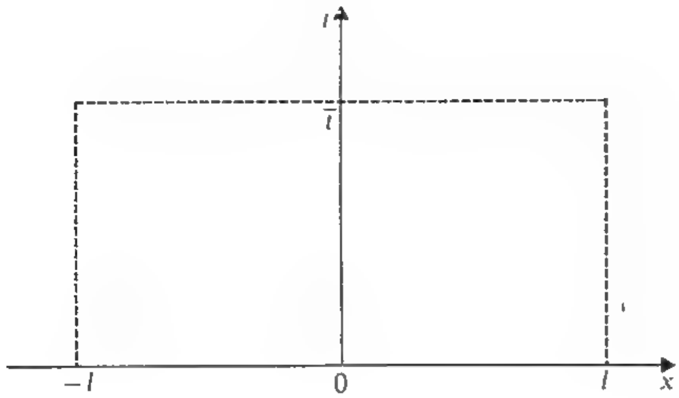
$$= \frac{f(x)}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-\eta^2} d\eta = f(x).$$

The solution gives the temperature distribution $u(x, t)$ in an infinite bar with a given initial temperature distribution $u(x, 0) = f(x)$ for $x \in R$.

Solution to SAQ 20

Extend the domain to $[-l, l] \times [0, \bar{t}]$, as shown in the figure, and let

$$u(x, t) = u(-x, t) \qquad (x, t) \in [-l, 0] \times [0, \bar{t}].$$



Then $x \longmapsto u(x, t)$ is an even function and is seen to be twice continuously differentiable over the extended domain, while at the same time

$$\frac{\partial u}{\partial x}(0, t) = 0.$$

In addition u satisfies the differential equation in the extended domain. By the maximum principle (in one dimension) the maximum must occur on either $t = 0$ or $x = \pm l$. Since $u(l, t) = u(-l, t)$ we have the required result that the maximum of u for $(x, t) \in [0, l] \times [0, \bar{t}]$ occurs at $t = 0$ or $x = l$.

3.6 APPENDIX: VECTORS

We have collected, in this appendix, some results about vectors which have appeared in previous courses and are relevant to this course. This appendix may be used for revision or reference. The results are presented in detail in *Unit M100 22, Linear Algebra I*, *Unit M201 1, Vector Spaces*, *Unit M201 16, Euclidean Spaces* and *Unit MST 282 1, Basic Tools.*

A **vector space** V over a *field F of* **scalars** is a set of elements with an internal binary operation $+: V \times V \longrightarrow V$ and an external binary operation of multiplication $F \times V \longrightarrow V$ satisfying the following axioms for all $\mathbf{a}, \mathbf{b}, \mathbf{c} \in V$ and $m, n \in F$:

1 $(\mathbf{a} + \mathbf{b}) + \mathbf{c} = \mathbf{a} + (\mathbf{b} + \mathbf{c})$

2 $\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$

3 $\exists \mathbf{0} \in V$ such that $\mathbf{0} + \mathbf{a} = \mathbf{a} \quad \forall \mathbf{a} \in V$

4 $\exists -\mathbf{a} \in V$ such that $\mathbf{a} + (-\mathbf{a}) = \mathbf{0}$

5 $m(\mathbf{a} + \mathbf{b}) = m\mathbf{a} + m\mathbf{b}$

6 $(m + n)\mathbf{a} = m\mathbf{a} + n\mathbf{a}$

7 $(mn)\mathbf{a} = m(n\mathbf{a})$

8 $1\mathbf{a} = \mathbf{a}$

When $F = R$ (the reals) we talk about a *real vector space*; when $F = C$ (the complex numbers) we have a *complex vector space.*

An important notion in a vector space is that of linear independence. A set $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ of vectors in V is **linearly independent** if

$$\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \cdots + \alpha_n\mathbf{v}_n = \mathbf{0} \qquad (\alpha_i \in F)$$

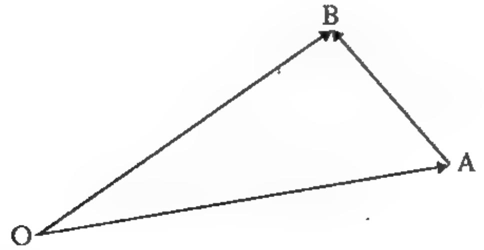
holds only for $\alpha_1 = \alpha_2 = \cdots = \alpha_n = 0$. Suppose now that $\{\mathbf{v}_1, \ldots, \mathbf{v}_m\}$ is a *maximal* linearly independent set in V, i.e. if another vector in V is included in the set we obtain a set which is linearly dependent. Then $\{\mathbf{v}_1, \ldots, \mathbf{v}_m\}$ is called a **basis** for V, and every other basis for V has precisely m vectors. We say that V has **dimension** m. It can also be shown that an arbitrary element $\mathbf{x} \in V$ can be expressed uniquely in the form

$$\mathbf{x} = x_1\mathbf{v}_1 + \cdots + x_n\mathbf{v}_n \qquad (x_i \in F).$$

In physical and engineering applications, it is useful to distinguish between physical quantities which are associated with a direction such as velocity, acceleration, force and magnetic field, and quantities with no associated direction such as mass, density and temperature. The latter can be represented by real numbers, while *geometric vectors* may be used to handle the former.

Geometric vectors are equivalence classes of parallel arrows of equal length. If $\overrightarrow{\mathrm{AB}}$ is an arrow, we write $\underrightarrow{\mathrm{AB}}$ for the geometric vector to which it belongs. The sum of two geometric vectors is obtained by choosing suitable arrows which can be added according to the *triangle law of addition*:

$$\underrightarrow{\mathrm{OA}} + \underrightarrow{\mathrm{AB}} = \underrightarrow{\mathrm{OB}}.$$



If λ is a real number then we define the product $\lambda\underrightarrow{\mathrm{AB}}$ to be the geometric vector whose arrows are parallel to those of $\underrightarrow{\mathrm{AB}}$ but with lengths multiplied by λ.

With respect to these definitions of addition and scalar multiplication the set G^3 of geometric vectors in space forms a real vector space. We often consider the set G^2 of geometric vectors in the plane, which is a *subspace* of this vector space. The null vector $\mathbf{0}$ is the geometric vector whose arrows are all of zero length.

Let $Oxyz$ be a frame of rectangular Cartesian coordinate axes and let $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ be geometric vectors whose arrows are of unit length and parallel to the coordinate axes Ox, Oy, Oz respectively. Then every geometric vector in the plane can be written uniquely in the form

$$\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j},$$

and every geometric vector in space can be written uniquely in the form

$$\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}.$$

Thus $\{\mathbf{i}, \mathbf{j}\}$ forms a basis for the space G^2 of geometric vectors in the plane, and $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$ forms a basis for G^3. a_1, a_2, a_3 are, respectively, the x-, y-, z-**coordinates** of $\mathbf{a}$.

Another important vector space is R^2, the set of ordered pairs of real numbers, with addition and scalar multiplication defined by

$$(a_1, a_2) + (b_1, b_2) = (a_1 + b_1, a_2 + b_2),$$

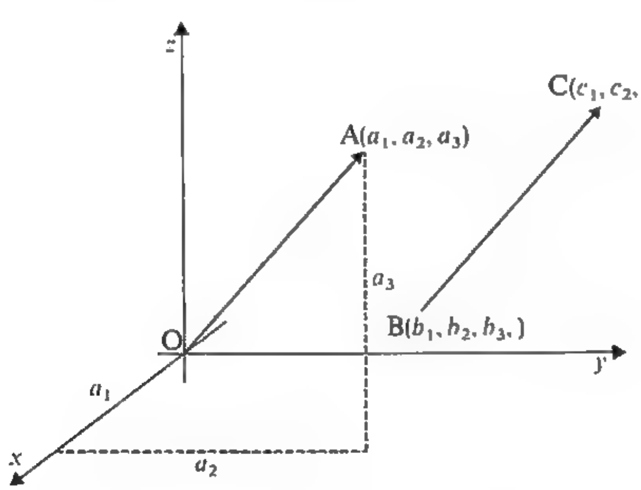
and

$$m(a_1, a_2) = (ma_1, ma_2).$$

R^2 is a real vector space with respect to these operations and we frequently identify R^2 with G^2 by the correspondence

$$a_1\mathbf{i} + a_2\mathbf{j} \leftrightarrow (a_1, a_2),$$

which is an *isomorphism*. Similarly we may identify R^3 with G^3. Under these identifications, $\underrightarrow{OA}$ (where O is the origin of coordinates) corresponds to the element of R^2 or R^3 which gives the coordinates of A in the coordinate system of $Oxyz$.



If $\overrightarrow{OA}$ and $\overrightarrow{BC}$ belong to the same geometric vector then

$$a_1 = c_1 - b_1, a_2 = c_2 - b_2, a_3 = c_3 - b_3,$$

where

$$\underrightarrow{OA} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k},$$

$$\underrightarrow{OB} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k},$$

$$\underrightarrow{OC} = c_1\mathbf{i} + c_2\mathbf{j} + c_3\mathbf{k}.$$

$\underrightarrow{OA}$ or (a_1, a_2, a_3) is the **position vector** of A.

An important property of a geometric vector is its *length* or *magnitude*. To define this more generally, we introduce the inner product.

A mapping $\cdot : V \times V \longrightarrow R$, where V is a real vector space, is called a (real) **inner product** if for all $\mathbf{a}, \mathbf{b}, \mathbf{c} \in V$ and $\lambda, \mu \in R$:

1 $\mathbf{a} \cdot \mathbf{a} = 0 \Leftrightarrow \mathbf{a} = \mathbf{0}$;

2 $\mathbf{a} \cdot \mathbf{a} \geqslant 0$;

3 $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$;

4 $\mathbf{a} \cdot (\lambda \mathbf{b} + \mu \mathbf{c}) = \lambda \mathbf{a} \cdot \mathbf{b} + \mu \mathbf{a} \cdot \mathbf{c}$.

We may specify an inner product on G^2 or G^3 by

$$\underrightarrow{OA} \cdot \underrightarrow{OB} = OA \,.\, OB \cos \theta,$$

where OA, OB denotes the length of the arrow $\overrightarrow{OA}$, $\overrightarrow{OB}$ respectively and θ is the angle between $\overrightarrow{OA}$ and $\overrightarrow{OB}$. The geometric vectors $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are *orthogonal*, i.e.

$$\mathbf{i} \cdot \mathbf{j} = \mathbf{j} \cdot \mathbf{k} = \mathbf{k} \cdot \mathbf{i} = 0.$$

(In fact, any two geometric vectors which are represented by perpendicular arrows are orthogonal.) Thus

$$\underrightarrow{OA} \cdot \underrightarrow{OB} = a_1 b_1 + a_2 b_2 + a_3 b_3$$

in terms of coordinates. Correspondingly, we *define*, for vectors in R^3,

$$(a_1, a_2, a_3) \cdot (b_1, b_2, b_3) = a_1 b_1 + a_1 b_2 + a_3 b_3.$$

This satisfies the axioms for an inner product on R^3.

The **norm** of a vector is defined by

$$\|\mathbf{a}\| = (\mathbf{a} \cdot \mathbf{a})^{\frac{1}{2}}.$$

Thus the norm of a geometric vector is just the length of an arrow representing it, and the norm of an element of R^3 is given by

$$\|a_1, a_2, a_3)\| = (a_1^2 + a_2^2 + a_3^2)^{\frac{1}{2}}.$$

We adopt the convention that a denotes $\|\mathbf{a}\|$, i.e. if a vector is denoted by a boldface symbol then the corresponding italic symbol denotes its norm. Note that $\mathbf{a} \cdot \mathbf{a} = a^2$.

An important theorem in real inner product spaces is *Schwarz's Inequality*:

$$(\mathbf{x} \cdot \mathbf{y})^2 \leqslant (\mathbf{x} \cdot \mathbf{x})(\mathbf{y} \cdot \mathbf{y}).$$

In G^3 (but not in G^2) we define the **vector product** $\times : G^3 \times G^3 \longrightarrow G^3$ by

$$\underrightarrow{OA} \times \underrightarrow{OB} = OA \,.\, OB \sin \theta\, \mathbf{n},$$

where θ is the angle $(0 \leqslant \theta < \pi)$ between $\overrightarrow{OA}$ and $\overrightarrow{OB}$, and $\mathbf{n}$ is the *unit vector* (i.e. vector whose norm is 1) orthogonal to $\underrightarrow{OA}$ and $\underrightarrow{OB}$, such that $\underrightarrow{OA}$, $\underrightarrow{OB}$ and $\mathbf{n}$ form a right-handed triad.

The vector product has the following properties:

1 $\mathbf{i} \times \mathbf{j} = \mathbf{k}, \mathbf{j} \times \mathbf{k} = \mathbf{i}, \mathbf{k} \times \mathbf{i} = \mathbf{j}, \mathbf{i} \times \mathbf{i} = \mathbf{j} \times \mathbf{j} = \mathbf{k} \times \mathbf{k} = \mathbf{0}$;

2 $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$;

3 $\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$;

4 $\mathbf{a} \times \mathbf{b} = (a_2 b_3 - a_3 b_2)\mathbf{i} + (a_3 b_1 - a_1 b_3)\mathbf{j} + (a_1 b_2 - a_2 b_1)\mathbf{k}$;

5 $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} \neq \mathbf{a} \times (\mathbf{b} \times \mathbf{c})$, in general.

Property 4 may be obtained by expanding

$$\mathbf{a} \times \mathbf{b} = (a_1 \mathbf{i} + a_2 \mathbf{j} + a_3 \mathbf{k}) \times (b_1 \mathbf{i} + b_2 \mathbf{j} + b_3 \mathbf{k}),$$

using Properties 1, 2 and 3 which are simple consequences of the definition.

If ABCD is a parallelogram, i.e.

$$\underset{\longrightarrow}{AB} = \underset{\longrightarrow}{DC} \quad \text{and} \quad \underset{\longrightarrow}{AD} = \underset{\longrightarrow}{BC},$$

then its *area* is given by

$$\|\underset{\longrightarrow}{AB} \times \underset{\longrightarrow}{BC}\|.$$

The scalar obtained by evaluating $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$ is called the **triple scalar product** of $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$. For convenience, we often write $\mathbf{a} \cdot \mathbf{b} \times \mathbf{c}$; no confusion should arise since $(\mathbf{a} \cdot \mathbf{b}) \times \mathbf{c}$ would be meaningless. We note that

$$\mathbf{a} \cdot \mathbf{b} \times \mathbf{c} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix},$$

and, by the property of determinants regarding interchange of columns, we obtain

$$\mathbf{a} \cdot \mathbf{b} \times \mathbf{c} = \mathbf{b} \cdot \mathbf{c} \times \mathbf{a} = \mathbf{c} \cdot \mathbf{a} \times \mathbf{b}.$$

The triple scalar product determines a function from $G^3 \times G^3 \times G^3 \longrightarrow R$.

We may show that the *volume* of a parallelepiped (i.e. a three-dimensional solid whose faces are parallelograms) is given by the modulus of the triple scalar product

$$|\mathbf{a} \cdot \mathbf{b} \times \mathbf{c}|,$$

where $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are geometric vectors represented by three arrows which generate the parallelepiped.

PARTIAL DIFFERENTIAL EQUATIONS OF APPLIED MATHEMATICS

1	*W*	The Wave Equation
2	*W*	Classification and Characteristics
3	*W*	Elliptic and Parabolic Equations
4		NO TEXT
5	*S*	Finite-Difference Methods I: Initial Value Problems
6	*W*	Fourier Series
7	*N*	Motion of Overhead Electric Train Wires
8	*S*	Finite-Difference Methods II: Stability
9	*W*	Green's Functions I: Ordinary Differential Equations
10	*W*	Green's Functions II: Partial Differential Equations
11	*S*	Finite-Difference Methods III: Boundary Value Problems
12		NO TEXT
13	*W*	Sturm-Liouville Theory
14	*W*	Bessel Functions
15	*N*	Finite-Difference Methods IV
16	*N*	Blood Flow in Arteries.

The letter after the unit number indicates the relevant set book; N indicates a unit not based on either book.

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